Heisenberg's uncertainty principle

We have seen situations where a function and its Fourier transform cannot be both localized. For example, the Fourier transform of a delta function (localized) is 1 (flat). Also, from Problem Set 3, we see if either f or $\hat{f}$ has compact support (localized), then the other *cannot* unless they are identically 0.

All these turn out to be examples of more general *uncertainty principles.* In the context of Fourier transform, they all roughly state that, in general, the more localized f is, the more wide spread $\hat{f}$ must be. The Heisenberg's uncertainty principle to be presented below is an example of the quantitative description of this general phenomena.

Suppose $f \in \mathcal{S}(\mathbb{R})$ with $\int |f|^2 dx = 1$. Then, one could think of the function $|f|^2$ as a probability density on $\mathbb{R}$. Plancherel's theorem tells us $\int |\hat{f}(\xi)|^2 d\xi = 1$, so $|\hat{f}|^2$ is another probability density on $\mathbb{R}$. The means of these densities are

$$m(f) = \int_{\mathbb{R}} x|f(x)|^2 dx, \qquad m(\hat{f}) = \int_{\mathbb{R}} \xi |\hat{f}(\xi)|^2 d\xi,$$

and their variances are

$$\mathcal{V}(f) = \int_{\mathbb{R}} |x - m(f)|^2 |f(x)|^2 dx, \qquad \mathcal{V}(\hat{f}) = \int_{\mathbb{R}} |\xi - m(\hat{f})|^2 |\hat{f}(\xi)|^2 \xi.$$

The variance measures the deviation of a distribution from its mean; the larger the variance is, the more spread-out the density is. Heisenberg's uncertainty principle states that

$$\mathcal{V}(f) \cdot \mathcal{V}(\hat{f}) \geq \frac{1}{16\pi^2},$$

with equality if and only if f is a normalized Gaussian function. The above lower bound indicates that f and $\hat{f}$ cannot be both too localized. To prove this inequality, let us assume for a moment that $m(f) = m(\hat{f}) = 0$, so we have

$$\begin{aligned} 4\pi^2 \Big(\int |x|^2 |f(x)|^2 dx \Big) \cdot \Big(\int |\xi|^2 |\hat{f}(\xi)|^2 d\xi \Big) &= \Big(\int |x|^2 |f(x)|^2 dx \Big) \cdot \Big(\int |f'(x)|^2 dx \Big) \\ &\geq \Big(\int |x f(x) f'(x)|^2 dx \Big)^2, \end{aligned}$$

where the second step follows from Cauchy-Schwarz inequality. Since the real part of $xf(x)f'(x)$ equals

$$\frac{x}{2}(\bar{f}f' + f\bar{f}') = \frac{x}{2}\frac{d}{dx}|f|^2,$$

it then follows that

$$4\pi^2 \mathcal{V}(f)\mathcal{V}(\hat{f}) \geq \frac{1}{4}\Big(\int x \frac{d}{dx} |f(x)|^2 dx \Big)^2 = \frac{1}{4}.$$

The general case with non-zero means could be reduced to the above special case by setting

$$F(x) = f(x + m(f)) e^{-2\pi i x \cdot m(\hat{f})}.$$

In order for the equality to hold, we must have equality when we apply Cauchy-Schwarz. This implies

$$f'(x) = \beta x \overline{f(x)}$$

for some complex constant β, and

$$f(x) = Ae^{-|\beta| x^2/2}$$

is the only square integrable solution, which in turn requires β to be negative real.