The isoperimetric problem

In this note, we give a simple but beautiful use of Fourier series to the following problem: among all simple closed curves in the plane with fixed perimeter, which one maximizes its enclosed area? If we fix the length of the simple closed curve γ to be ℓ, then it turns out that the area $\mathcal{A}$ enclosed by the curve satisfies the inequality

$$\mathcal{A} \leq \frac{\ell^2}{4\pi}, \tag{1}$$

with equality if and only if γ is a circle. The solution we give below is due to Hurwitz (1901).

To start, we first note that it suffices to prove the statement for $\ell = 1$, and all other values of ℓ will follow by scaling. Thus, we may assume $\gamma : [0, 1] \to \mathbb{R}^2$ has unit length 1, and we parametrize it at unit speed so that

$$|\gamma'(t)| = x'(t)^2 + y'(t)^2 \equiv 1.$$

In particular, this parametrization implies

$$\int_0^1 (x'(t)^2 + y'(t)^2)dt = 1. \tag{2}$$

Also, in order to avoid technical issues, we look for curves γ whose unit tangent directions (x', y') are continuous under unit speed parametrization, although this requirement turns out to be unnecessary for the main result to hold.

The area $\mathcal{A}$ enclosed by γ could be represented by

$$\mathcal{A} = \frac{1}{2}\left| \int_0^1 \big(x(t)y'(t) - x'(t)y(t)\big)dt \right|. \tag{3}$$

Since γ is closed, its components x and y are both continuous and periodic with period 1, so we can consider their Fourier series with coefficients

$$a_n = \int_0^1 x(t)e^{-2\pi int}dt \qquad \text{and} \qquad b_n = \int_0^1 y(t)e^{-2\pi int}dt.$$

The Fourier coefficients of x' and y' are thus $2\pi ina_n$ and $2\pi inb_n$, respectively. An application of Parseval's identity to (2) gives the relation

$$\sum_{n\in\mathbb{Z}} n^2(|a_n|^2 + |b_n|^2) = \frac{1}{4\pi^2}. \tag{4}$$

We also apply Parseval's identity to (3) to get

$$\mathcal{A} = \frac{1}{2}\left| \int_0^1 \big(x(t)y'(t) - x'(t)y(t)\big)dt \right| = 2\pi \left| \sum_{n\in\mathbb{Z}} na_n\overline{b_n} \right|,$$

Now, we observe that for any integer n, we have

$$2|n||a_n\overline{b_n}| \leq 2n^2|a_n||b_n| \leq n^2(|a_n|^2 + |b_n|^2), \tag{5}$$

so inserting (4) into the area expression above gives

$$\mathcal{A} \le \frac{1}{4\pi}.$$

We now turn to the second part of the problem, finding the circumstances in which an equality holds. A closer look at the above proof reveals that the only two place we used an inequality is in (5) and also taking the sum out of the absolute value sign. In fact, in order for equality in (1) to hold, the inequalities in both places must be equalities.

We first investigate (5) for each n. It is clear that in this case, we must have $a_n = b_n = 0$ for all $|n| \ge 2$, so x and y has the form

$$x(t) = a_{-1}e^{-2\pi it} + a_0 + a_1e^{2\pi it}, \qquad y(t) = b_{-1}e^{-2\pi it} + b_0 + b_1e^{2\pi it}.$$

An equality in (5) thus gives

$$2|a_1\overline{b_1}| = |a_1|^2 + |b_1|^2,$$

which, together with the fact that $a_{-1} = \overline{a_1}$ and $b_{-1} = \overline{b_1}$, as well as the relation (4), imply

$$|a_1| = |b_1| = \frac{1}{4\pi}.$$

We could thus write

$$a_1 = \frac{1}{4\pi}e^{2\pi i\alpha}, \qquad b_1 = \frac{1}{4\pi}e^{2\pi i\beta}.$$

Now we look at the second place where we have used an inequality: taking the sum out of the absolute value sign. The above expressions of a_1 and b_1 suggests that $\mathcal{A}$ could be written as

$$\mathcal{A} = 2\pi|a_1\overline{b_1} - a_{-1}\overline{b_{-1}}| = 2\pi|a_1\overline{b_1} - \overline{a_1}b_1|.$$

In order for $\mathcal{A} = \frac{1}{4\pi}$ to hold, we should have $|a_1\overline{b_1} - \overline{a_1}b_1| = \frac{1}{8\pi^2}$, which in turn suggests that $\alpha - \beta = \frac{2k+1}{4}$. Inserting this into the expression of x and y, we finally get

$$x(t) = a_0 + \frac{1}{2\pi}\cos\big(2\pi(\alpha + t)\big), \qquad y(t) = b_0 \pm \frac{1}{2\pi}\sin\big(2\pi(\alpha + t)\big),$$

where the sign in y depends on whether k is odd or even. This is nothing but a circle centered at (a_0, b_0) with radius $\frac{1}{2\pi}$. The value of α and the sign in front of the sine component in $y(\cdot)$ represent the starting location and the orientation of the parametrization. This completes the proof.