L^2 convergence of Fourier series

In this note, we will discuss the convergence of Fourier series from a completely different viewpoint: convergence in L^2. The modern setting here is that $L^2(\mathbb{T})$ is a Hilbert space. In the end, we will prove that the functions $\{e_n\}_{n\in\mathbb{Z}}$ defined by

$$e_n(x) = e^{2\pi inx}, \qquad x \in \mathbb{T}$$

is an orthonormal basis of $L^2(\mathbb{T})$, and that if $f \in L^2(\mathbb{T})$, then we have

$$\|f\|_{L^2}^2 = \sum_{n\in\mathbb{Z}} |a_n|^2.$$

The latter is the so called Parseval's identity. We first recall some basic facts about Hilbert spaces.

The concept of a Hilbert space is a natural generalization of that of a Euclidean space. Here, the dot product is replaced by the notion of an *inner product*. For a (complex) vector space V, a rule

$$\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}, \qquad (f, g) \to \langle f, g\rangle$$

is called an inner product if the followings are satisfied:

1. $\langle\cdot,\cdot\rangle$ is linear in its first argument; that is,

$$\langle af_1 + bf_2, g\rangle = a\langle f_1, g\rangle + b\langle f_2, g\rangle.$$

2. For any f and g, we have

$$\langle f, g\rangle = \overline{\langle g, f\rangle}.$$

This (together with the first rule) imply that the inner product must be anti-linear in its second argument, and that $\langle f, f\rangle$ is real.

3. For any $f \neq 0$, we have $\langle f, f\rangle > 0$.

We say two vectors f and g are orthogonal to each other, denoted by $f \perp g$, if $\langle f, g\rangle = 0$. Note that this is consistent with the finite dimensional case with dot products.

The first thing coming with an inner product is the *Cauchy-Schwarz inequality*, which asserts that if $\langle\cdot,\cdot\rangle$ is an inner product, then we have

$$|\langle f, g\rangle| \leq \langle f, f\rangle^{\frac{1}{2}} \langle g, g\rangle^{\frac{1}{2}},$$

and equality holds if and only if $f = \lambda g$ for some $\lambda \in \mathbb{C}$.

A vector space V with an inner product structure on it is called an inner product space. There is a natural notion of norm generated by the inner product, namely

$$\|f\| := \langle f, f\rangle^{\frac{1}{2}},$$

and the metric on V is thus naturally defined by

$$d(f, g) = \|f - g\|.$$

One can easily check that this indeed satisfies all the properties of a metric. In addition, it turns out that a norm $\| \cdot \|$ on V is induced by an inner product if and only if the parallelogram law

$$\|f+g\|^2 + \|f-g\|^2 = 2\big(\|f\|^2 + \|g\|^2\big)$$

holds for all $f, g \in V$. The 'only if' part follows immediately from the rules of an inner product, while the 'if' part is the content of the theorem of Jordan and von Neumann.

We are now ready to introduce the concept of a Hilbert space. A Hilbert space $\mathcal{H}$ is a (complex) inner product space which is also a *complete* metric space with metric generated by the inner product. The most important fact about Hilbert spaces is the Riesz representation theorem.

Theorem 1 (**Riesz representation**). *Every continuous linear functional on $\mathcal{H}$ is given by the inner product by some element in $\mathcal{H}$. More precisely, for every continuous linear $\ell : \mathcal{H} \to \mathbb{C}$, there exists $g \in \mathcal{H}$ such that*

$$\ell(f) = \langle f, g \rangle$$

for every $f \in \mathcal{H}$.

Euclidean spaces are finite dimensional Hilbert spaces with dot product being the inner product. A particularly interesting concrete infinite dimensional Hilbert space is the space $\ell^2(\mathbb{Z})$, consisting the infinite sequences

$$\mathbf{a} = (\cdots, a_{-2}, a_{-1}, a_0, a_1, a_2, \cdots)$$

such that $\sum_n |a_n|^2 < +\infty$. The inner product is given by

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{n \in \mathbb{Z}} a_n \overline{b_n},$$

and Cauchy-Schwarz inequality guarantees that the right hand side is finite. The space $L^2(\mathbb{T})$ is also a Hilbert space with

$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}dx.$$

For both of the examples, the rules of the inner product are straightforward to check, but *completeness* is the main thing.

Finally, just like in the finite dimensional case, a Hilbert space $\mathcal{H}$ also comes with the concept of a basis. Given a set of vectors $\{v_n\}$, a natural generalization of the definition of a basis is that any vector $f \in \mathcal{H}$ could be written as a linear combination of the v_n's as $f = \sum c_n v_n$. However, this involves an infinite sum, and there would be an issue with convergence. Indeed, the meaning of the equal sign should be interpreted as 'any vector f can be approximated arbitrarily well by finite linear combinations of the v_n's. ' Thus, we say $\{v_n\}$ is a basis for $\mathcal{H}$ if for any $\epsilon > 0$, there exists a finite set $I \in \mathbb{Z}$ and coefficients c_n's such that

$$\|f - \sum_{n \in I} c_n v_n\| < \epsilon.$$

We now come back to the main business, proving that the set

$$e_n(x) = e^{2\pi inx}$$

is an orthonormal basis of $L^2(\mathbb{T})$. It is easy to see that they are *orthonormal* in the sense that

$$\langle e_n, e_m \rangle = \int_0^1 e^{2\pi inx} e^{-2\pi imx} dx = \delta_{n,m},$$

where $\delta_{n,m} = 1$ if $n = m$ and is 0 if $n \neq m$. To show $\{e_n\}$ is also a basis, we need to check that for any $f \in L^2(\mathbb{T})$, there exists a set of complex numbers $\{a_n\}$ such that $\sum_{|n|\leq N} a_n e_n$ converges to f in L^2. The obvious choice of these coefficients are

$$a_n = \langle f, e_n \rangle = \int_0^1 f(x) e^{-2\pi inx} dx,$$

as the Fourier coefficient a_n is nothing but representing the component of f on the direction e_n. Since smooth functions on $\mathbb{T}$ are dense in $L^2(\mathbb{T})$, for any $f \in L^2(\mathbb{T})$, there exists a smooth φ on $\mathbb{T}$ such that

$$\|f - \varphi\|_{L^2} < \epsilon.$$

Thus, with the usual notation $S_N f = \sum_{|n|\leq N} a_n e_n$, we have

$$\|f - S_N f\|_{L^2} \leq \|f - \varphi\|_{L^2} + \|\varphi - S_N \varphi\|_{L^2} + \|S_N(f - \varphi)\|_{L^2},$$

with the obvious meaning that $S_N \varphi$ and $S_N(f - \varphi)$ are represented by the corresponding Fourier coefficients of φ and $f - \varphi$. Now, the first term above is bounded by ϵ by the choice of φ. For the third term, since

$$S_N g \perp (g - S_N g),$$

this implies

$$\|S_N g\|_{L^2}^2 = \|g\|_{L^2}^2 - \|g - S_N g\|_{L^2}^2 \leq \|g\|_{L^2}^2$$

by the Pythagorean theorem. Thus, setting $g = f - \varphi$ gives that the third term is also smaller than ϵ. Finally, since φ is smooth, $S_N \varphi$ converges uniformly to φ, so the second term can also be made smaller than ϵ for all large enough N. We have thus proved the following.

Theorem 2. *The set $\{e_n\}$ is an orthonormal basis of $L^2(\mathbb{T})$.*

As a consequence, we also have $\|S_N f\|_{L^2}^2 \to \|f\|_{L^2}^2$, where the former is given by

$$\|S_N f\|_{L^2} = \sum_{|n|\leq N} |a_n|^2.$$

This immediately gives

$$\|f\|_{L^2}^2 = \sum_{n\in\mathbb{Z}} |a_n|^2. \tag{1}$$

This is a special case of Parseval's identity. In fact, it only depends on the orthonormality and completeness of $\{e_n\}$. If $\{h_n\}$ is another set of orthonormal basis and $c_n = \langle f, h_n \rangle$, then identity (1) also holds for the set $\{c_n\}$. Of course, if any of the c_n's is missing, then the equality becomes an inequality. This is called Bessel's inequality.

Corollary 3 (Bessel). *Let* $\{h_n\}$ *be an orthonormal system of* $\mathcal{H}$ *(not necessarily a basis) and* $c_n = \langle f, h_n \rangle$*. Then we have*

$$\sum_{n\in\mathbb{Z}} |c_n|^2 \leq \|f\|_{L^2}^2.$$

Now suppose g also belongs to $L^2(\mathbb{T})$ with Fourier coefficients $\{b_n\}$. Using the identity

$$\langle f, g \rangle = \frac{1}{4}\big[\|f+g\|^2 - \|f-g\|^2 + i\big(\|f+ig\|^2 - \|f-ig\|^2\big)\big]$$

as well as (1), we immediately get the following Parseval's identity.

Proposition 4 (Parseval). *We have*

$$\langle f, g \rangle = \sum_{n\in\mathbb{Z}} a_n \overline{b_n}.$$

Finally, if $f \in L^2(\mathbb{T})$, then by Parseval's identity, the set of its Fourier coefficients $\{a_n\}$ belongs to $\ell^2(\mathbb{Z})$ with $\|f\|_{L^2} = \|\mathbf{a}\|_{\ell^2}$. Conversely, if $\{a_n\}$ is a sequence in $\ell^2(\mathbb{Z})$, then the series

$$\sum_{n\in\mathbb{Z}} a_n e_n$$

defines an element in $f \in L^2(\mathbb{T})$ with equal norms. We thus have the following proposition.

Proposition 5. $L^2(\mathbb{T})$ *is isomorphic to* $\ell^2(\mathbb{Z})$*.*

In fact, all infinite dimensional separable Hilbert spaces can be identified with $\ell^2(\mathbb{Z})$. But this is irrelevant here, so we will not discuss in detail.