Solutions to the midterm

Problem 1. (a) $\sigma_n f \geq 0$ as it is the convolution with the Fejér kernel F_n, which is positive. For $S_n f$, as the Dirichlet kernel D_n is not always positive, it is not true. For example, one can take f such that it is positive at the points where D_n is negative, and 0 at where D_n is positive, then

$$(S_n f)(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) D_n(-x) dx = \int f(x) D_n(x) dx < 0.$$

(b) Since the Cesàro sum of the Fourier series of a continuous function uniformly converges to f, the sequence $\{s_n\}$ has Cesàro limit $f(0)$. Since a_n is positive, we thus conclude that the series $\sum a_n$ itself sums to $f(0)$.

(c) A simple application of Plancherel's theorem; the answer is π.

(d) $f(x) = 1 + \frac{1}{2}(e^{2\pi i x} + e^{-2\pi i x})$, so $\hat{f}(0) = 1$, $\hat{f}(1) = \hat{f}(-1) = \frac{1}{2}$, and $\hat{f}(n) = 0$ for all other n. As for the Fourier coefficients of f_k, we have

$$\hat{f}_k(n) = \left[\hat{f}(n)\right]^k,$$

so it easily follows that the limit is 1.

Problem 2. (a) The equation can be solved by odd extension to the whole real line, and then restricting the solution to positive x.

(b) The nontrivial part here is the uniform convergence of $u(t, \cdot)$ to f. Indeed, one needs f to be uniformly continuous on the real line, but this is true as long as f is continuous and vanishes at infinity.

Problem 3. (a) $\hat{f}$ has compact support in $(-\frac{1}{2}, \frac{1}{2})$, so it can be extended to a smooth periodic function on the whole real line, thus having an absolutely convergent Fourier series. The Fourier coefficients of $\hat{f}$ are

$$a_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{f}(\xi) e^{-2\pi i \xi n} d\xi = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi (-n)} d\xi = f(-n).$$

Thus, by Plancherel and then Parseval, we have

$$\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{f}(\xi)|^2 d\xi = \sum_n |f(-n)|^2 = \sum_n |f(n)|^2.$$

(b) We write $\hat{f}$ in terms of its Fourier series as

$$\hat{f}(\xi) = \sum_n a_n e^{2\pi i n \xi} = \sum_n f(-n) e^{2\pi i n \xi} = \sum_n f(n) e^{-2\pi i n \xi},$$

and substitute into the inversion formula

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} d\xi = \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{f}(\xi) e^{2\pi i \xi x} dx.$$

The answer should follow immediately. Everything is nicely convergent here as $f \in \mathcal{S}(\mathbb{R})$.

(c) If we let $f_\lambda(x) = f(x/\lambda)$, then $\widehat{f_\lambda}$ has compact support in $(-\frac{1}{2}, \frac{1}{2})$, and we can use part(b).

Problem 4. The form of the density ρ is relatively easy to guess. In fact, if ρ is the density, then for any continuous function f on the circle, we should have

$$\frac{1}{N} \sum_{n=1}^{N} f(\xi_n) \to \int_0^1 f(x)\rho(x)dx.$$

Taking f to be the trigonometric functions $e^{2\pi i k x}$, we see that the Fourier coefficients of ρ (viewed as a function periodically extended to the whole real line) are

$$a_0 = 1, a_1 = a_{-1} = \frac{1}{2},$$

and all others are 0. Thus, ρ should have the expression

$$\rho(x) = 1 + \frac{1}{2}(e^{2\pi i x} + e^{-2\pi i x}) = 1 + \cos(2\pi x).$$

Once you have got this expression, the proof just follows in exactly the same way as in the class.

Problem 5. (a) First of all, we observe that the limit f must be continuous. Thus, for any $\epsilon > 0$, we choose δ such that

$$\sup_n |f_n(x) - f_n(y)| < \epsilon, \qquad |f(x) - f(y)| < \epsilon$$

whenever $|x - y| < \delta$. Now, we choose points $y_1, \cdots, y_K$ on the circle such that the union of their δ-neighborhoods covers the circle. Since f_n converges to f pointwise and there are only finitely many y_j's, there exists N large enough such that for all $n > N$, we have

$$\sup_j |f_n(y_j) - f(y_j)| < \epsilon.$$

We should note that N here depends on ϵ (through δ) *only*. Thus, for any $xin\mathbb{T}$, there is some y_j from the above K points such that $|x - y_j| < \epsilon$, and

$$|f_n(x) - f(x)| \leq |f_n(x) - f_n(y_j)| + |f_n(y_j) - f(y_j)| + |f(y_j) - f(x)|.$$

The first and third terms are smaller than ϵ because $|x - y_j| < \delta$. The second term is smaller than ϵ when $n > N$. Since N only depends on ϵ but NOT x, we have proved that

$$f_n \to f$$

uniformly on $\mathbb{T}$.

(b) The same as in class.

(c) This is the hardest part in the midterm. We first write

$$g_n(x) = f(x) - (S_n f)(x).$$

Part (b) shows that g_n converges pointwise to 0. Furthermore, since $f \in \mathcal{C}^\alpha$, the equicontinuity of g automatically implies the equicontinuity of $S_n f$, so it suffices to show the sequence $\{g_n\}$ is equicontinuous. By definition, g_n has the expression

$$g_n(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} (f(x) - f(x-z)) D_n(z) dz,$$

and hence

$$g_n(x) - g_n(y) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \big(f(x) - f(x-z) - f(y) + f(y-z)\big) D_n(z) dz. \tag{0.1}$$

We now seek a bound on $|g_n(x) - g_n(y)|$ that is *independent* of n. First, we notice the Dirichlet kernel can be bounded by

$$|D_n(z)| = \left| \frac{\sin((2n+1)\pi z)}{\sin(\pi z)} \right| < \frac{C}{|z|}$$

on $[-\frac{1}{2}, \frac{1}{2}]$. Then, we observe that the other part of the integrand can be bounded by

$$\begin{aligned} |f(x) - f(x-z) - f(y) + f(y-z)| &\leq |f(x) - f(x-z)| + |f(y) - f(y-z)| \\ &\leq C|z|^\alpha, \end{aligned}$$

as well as

$$\begin{aligned} |f(x) - f(x-z) - f(y) + f(y-z)| &\leq |f(x) - f(y)| + |f(x-z) - f(y-z)| \\ &\leq C|x-y|^\alpha. \end{aligned}$$

Thus, we obtain the bound

$$|f(x) - f(x-z) - f(y) + f(y-z)| < C\big(|x-y|^\alpha \wedge |z|^\alpha\big),$$

uniformly in x, y and z. As a consequence, (0.1) could then be written as

$$\begin{aligned} |g_n(x) - g_n(y)| &\leq C \int_{|z|<\frac{1}{2}} \big(|x-y|^\alpha \wedge |z|^\alpha\big) \cdot |z|^{-1} dz \\ &= C\bigg(\int_{|z|<|x-y|} |z|^{-1+\alpha} dz + \int_{|x-y| \leq |z| < \frac{1}{2}} \frac{|x-y|^\alpha}{|z|} dz \bigg) \\ &\leq C\big(|x-y|^\alpha + |x-y|^\alpha \log(1/|x-y|^\alpha)\big) \\ &< C|x-y|^\alpha \log(1/|x-y|). \end{aligned}$$

This shows the equicontinuity of $\{g_n\}$.