Midterm Assignment

MA433, Autumn term 2014

This assignment is due in class on Thursday, November 6th. There are 60 points in total. Problems 1 and 2 are compulsory. Choose two problems from the remaining three. If you do all the three, then your two best answers will be counted.

Problem 1. Your answers to these questions should be short.

(a) [5] Let $f \in \mathcal{C}(\mathbb{T})$ with $f \geq 0$. Is it true that $(\sigma_n f)(x) \geq 0$ for all n and x? What about $S_n f$? Prove or give counter examples (in the latter case an explanation of how the function looks like would be sufficient).

(b) [5] Let $f \in \mathcal{C}(\mathbb{T})$. Suppose its Fourier coefficients satisfy $a_n \geq 0$ for every n. Show that the Fourier series converges absolutely with $\sum_n |a_n| = f(0)$. You could use the fact that the sum of a positive series equals its Cesàro sum (though possibly infinity).

(c) [5] Let f be the indicator function of the interval $[-1, 1] \subset \mathbb{R}$. Compute the Fourier transform of f, and use it to evaluate the integral $\int_{\mathbb{R}} \frac{\sin^2 x}{x^2} dx$.

(d) [5] Let $f(x) = 1 + \cos 2\pi x$, and let

$$f_k = \underbrace{f * \cdots * f}_{k \text{ times}}.$$

What is the value of $\lim_{k \to +\infty} f_k(1/2)$?

Problem 2 (Heat flow on the positive half real line). Consider the equation

$$\begin{cases} \partial_t u = \partial_x^2 u \\ u(0, x) = f(x) \\ u(t, 0) = 0, \quad \forall t \geq 0 \end{cases} . \tag{0.1}$$

Here, the unknown function u is defined on $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^+$. The initial temperature f is smooth with $f(0) = 0$ and

$$f(x) \to 0$$

as $x \to +\infty$, as fast as you wish. This equation models the heat diffusion on a semi-infinite rod with the temperature at one end ($x = 0$) being held at 0.

(a) [5] Find an expression of the solution u in terms of f.

(b) [5] Prove your expression of u satisfies the above heat equation with $u(t, 0) = 0$ and

$$u(t, \cdot) \to f$$

uniformly as $t \to 0$. (You could freely interchange the differentiation with integration, though you should convince yourself this is allowed here.)

Problem 3 (**Shannon sampling theorem**). Let $f \in \mathcal{S}(\mathbb{R})$, so $\hat{f}$ is smooth. We further suppose $\hat{f}$ has compact support contained in the interval $(-\frac{1}{2}, \frac{1}{2})$. It then turns out that, in this case, the knowledge of $f(n)$ at integer values $n \in \mathbb{Z}$ completely determines f: if $g \in \mathcal{S}(\mathbb{R})$ is another function with $\hat{g}$ supported in the interval $(-\frac{1}{2}, \frac{1}{2})$ and $f(n) = g(n)$ for all $n \in \mathbb{Z}$, then $f = g$. This is a fundamental theorem in information theory.

(a) [5] Prove that

$$\int_{\mathbb{R}} |f(x)|^2 dx = \sum_{n \in \mathbb{Z}} |f(n)|^2.$$

(b) [5] Show that

$$f(x) = \sum_{n \in \mathbb{Z}} f(n) \cdot \frac{\sin \pi(x-n)}{\pi(x-n)}.$$

This precisely says the knowledge of f on integers completely determines f.

(c) [5] Now suppose $\hat{f}$ is compactly supported in the interval $(-\frac{\lambda}{2}, \frac{\lambda}{2})$. Show that

$$f(x) = \sum_{n \in \mathbb{Z}} f(n/\lambda) \cdot \frac{\sin\left(\pi\lambda(x - n/\lambda)\right)}{\pi\lambda(x - n/\lambda)}.$$

Remark: Part(c) is a version of Shannon's sampling theorem. What does it mean?

Problem 4. [15] This problem investigates other asymptotic distributions on $[0, 1]$. For a sequence $\{\xi_n\} \in [0, 1]$, we say it has asymptotic distribution with density ρ if

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[a,b]}(\xi_n) = \int_a^b \rho(x) dx$$

for every $[a, b] \subset [0, 1]$. Now, suppose $\{\xi_n\}$ is a sequence of numbers in the interval $[0, 1]$ such that for every $k \neq 0, \pm 1$, we have

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k \xi_n} \to 0,$$

while for $|k| = 1$, we have

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \xi_n} \to \frac{1}{2}, \qquad \frac{1}{N} \sum_{n=1}^{N} e^{-2\pi i \xi_n} \to \frac{1}{2}.$$

Write down the density ρ of the asymptotic distribution of $\{\xi_n\}$, and prove your answer. (Hint: an important step in showing the equidistribution is to show

$$\frac{1}{N} \sum_{n=1}^{N} f(\xi_n) \to \int_0^1 f(x) dx \tag{0.2}$$

for every continuous function f on the circle. How should (0.2) be changed here?)

Problem 5. A function belongs to $\mathcal{C}^\alpha$ if there exists $C > 0$ such that $|f(x) - f(y)| < C|x-y|^\alpha$ for all x, y. Dirichlet's theorem implies the Fourier series of $\mathcal{C}^\alpha(\mathbb{T})$ functions converge pointwise. The purpose of this problem is to show that the convergence is actually *uniform*. (Also recall from PS 2 that the series converges absolutely if $\alpha > \frac{1}{2}$).

To prove uniform convergence, we need the following notion of *equicontinuity*: a sequence of functions $\{f_n\}$ defined on an interval $[a, b]$ is said to be an *equicontinuous* family if for every $\epsilon > 0$, there exists δ such that

$$\sup_n |f_n(x) - f_n(y)| < \epsilon$$

whenever $|x - y| < \delta$.

(a) [5] Show that if $\{f_n\}$ is an equicontinuous family on $\mathbb{T}$ and $f_n(x) \to f(x)$ for every x, then f_n converges to f uniformly.

(b) [5] Let $f \in \mathcal{C}^\alpha(\mathbb{T})$ with $\alpha > 0$. Prove $(S_n f)(x) \to f(x)$ for all x.

(c) [5] Show that $\{S_n f\}$ is an equicontinuous family (under the assumption $f \in \mathcal{C}^\alpha$), and conclude $S_n f \to f$ uniformly.