Heat and wave equations on the circle/interval

These are two very different equations, corresponding respectively to PDEs of *parabolic* type and *hyperbolic* type. We put them in the same note in order to compare their distinguishing features.

1 Heat equation

Consider the heat equation on the circle

$$\partial_t u = \partial_x^2 u. \tag{1.1}$$

Here, $u(t,x)$ describes the temperature at time t and location x, and we suppose $f(x) = u(0,x)$ is the initial temperature at time 0. We assume f is continuous on $\mathbb{T}$, and it is natural to expect that the whole process u is determined once the initial data f is given. At a formal level, we can expand $u(t,\cdot)$ into its Fourier series as

$$u(t,x) = \sum_{n\in\mathbb{Z}} A_n(t) e^{2\pi i n x}.$$

Since u solves the equation (1.1) together with initial data f, the coefficients $A_n(t)$'s should have the form

$$A_n(t) = a_n e^{-4\pi^2 n^2 t}, \qquad a_n = \int_0^1 f(x) e^{-2\pi i n x} dx.$$

Thus, a simple calculation suggests we could write the solution u as

$$u(t,x) = \int_0^1 p_t(y) f(x-y) dy = (p_t * f)(x),$$

where

$$p_t(x) = \sum_{n\in\mathbb{Z}} e^{-4\pi^2 n^2 t} e^{2\pi i n x} \tag{1.2}$$

is the *heat kernel* on the circle. The above derivation is only formal since the interchange(s) between the differentiation, the infinite sum and integral have not been justified yet, but our claim is that, $u = p_t * f$ is the only function with the following properties:

1. u solves the heat equation (1.1).
2. u is infinitely differentiable in both $t > 0$ and x.
3. The solutions $u(t,\cdot)$ uniformly converges to f as $t \to 0$.

That $u = p_t * f$ is the *only* solution with the above properties is easy to show. In fact, if u is a solution, then it must satisfy all the calculations we have made above, and hence $u = p_t * f$. So we only need to show that it is indeed a solution.

First of all, since for any $t > 0$, the infinite sum defining p_t has a Gaussian (rapid) decay, one can just differentiate it term-wise inside the summation, and conclude that u

is smooth and solves (1.1). It thus remains to prove the third property above. In fact, we have

$$|u(t,x) - f(x)| = \big|\sum_n a_n (e^{-4\pi^2 n^2 t} - 1) e^{2\pi i n x}\big| \leq \sum_{n\in\mathbb{Z}} |a_n| (1 - e^{-4\pi^2 n^2 t}).$$

Ideally, since for each n, $e^{-4\pi^2 n^2 t} \to 0$ as $t \to 1$, one would hope the uniform convergence just follows from that. However, things are not that simple as it is not always valid to change the limit with infinite sum. Nevertheless, if a_n's decay fast, then the change of limits will be valid, and this is precisely the case when f is regular enough.

Let us suppose $f \in \mathcal{C}^1(\mathbb{T})$ so that $\sum_n |a_n|$ is absolutely convergent. For any $\epsilon > 0$, we write

$$\sum_{n\in\mathbb{Z}} |a_n| (1 - e^{-4\pi^2 n^2 t}) = \sum_{|n|\leq N} |a_n| (1 - e^{-4\pi^2 n^2 t}) + \sum_{|n|>N} |a_n| (1 - e^{-4\pi^2 n^2 t}),$$

where N is chosen such that $\sum_{|n|>N} |a_n| < \epsilon$. This gives the bound for the second term above. For the first one, since the sum is now finite, we can send $t \to 0$ so that this term is also smaller than ϵ. We have thus proved that

$$\lim_{t\to 0} \sup_x |u(t,x) - f(x)| = 0$$

if $f \in \mathcal{C}^1(\mathbb{T})$. For general continuous f such that $\sum_n |a_n|$ diverges, the above argument does not work, and we need more detailed information about the kernel p_t. The first thing we need is

$$\int_0^1 p_t(x) dx = 1$$

for any positive t. This is obvious from the form of p_t in (1.2). However, what is *less obvious* is that we actually have

$$p_t(x) = (4\pi t)^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} e^{-\frac{(x+n)^2}{4t}}. \tag{1.3}$$

We will not be able to prove this expression until several weeks later, but if you *trust me* for this moment, you can easily see that

$$p_t(x) > 0 \tag{1.4}$$

for every t and x. This (strict) positivity and the obvious fact that p_t integrates to 1 will be all we need to extend the uniform convergence to general continuous f on the circle.

To continue, let us first note that

$$|u(t,x)| = \left| \int_0^1 p_t(x-y) f(y) dy \right| \leq \|f\|_\infty \int_0^1 |p_t(x)| dx = \|f\|_\infty,$$

which immediately gives $\|p_t * f\|_\infty \leq \|f\|_\infty$ for any $f \in \mathcal{C}(\mathbb{T})$. Now, fix $f \in \mathcal{C}(\mathbb{T})$, choose $g \in \mathcal{C}^1(\mathbb{T})$ such that $\|g - f\|_\infty < \epsilon$, and we write

$$\|p_t * f - f\|_\infty \leq \|p_t * f - p_t * g\|_\infty + \|p_t * g - g\|_\infty + \|g - f\|_\infty.$$

The third term is smaller than ϵ by the choice of g, so is the first term since $\|p_t * (f - g)\|_\infty \leq \|f - g\|_\infty$. For the second term, since $g \in \mathcal{C}^1(\mathbb{T})$, it will also be smaller than ϵ for all small t. We have thus proved that

$$p_t * f \to f \qquad \text{uniformly} \quad \text{as } t \to 0$$

for all $f \in \mathcal{C}(\mathbb{T})$.

In fact, a more careful analysis of p_t will give that for any $\delta > 0$, we have

$$\int_{\delta < |x| \leq \frac{1}{2}} |p_t(x)| dx \to 0,$$

as $t \to 0$. This suggests the behavior of the heat kernel p_t as $t \to 0$ is *very similar* to the behavior of the Fejér kernel F_n as $n \to 0$. One should indeed think of the convergence $\|p_t * f - f\|_\infty \to 0$ as an analogy to that $\|\sigma_n f - f\|_\infty \to 0$ for continuous function f on the circle. The verification of these properties of p_t will be left as an exercise. We will also give a more systematic study of these 'good' kernels and the convergences they carry later in this course.

Finally, we end the discussion of heat equation by summarizing two properties that can be seen from above.

1. The strict positivity of p_t (1.4) implies that, as long as $f \geq 0$ and not identically 0, then no matter how small the time t is, $u(t, x)$ will be positive *everywhere*. This is the *infinite propagation* property of the heat flow.

2. The heat flow *cannot be reversed* in the sense that given the temperature u at time t, one cannot solve the heat equation backwards.

2 Wave equation

Think of a string as stretched in the interval $(0, 1)$, but tied at its two end points $x = 0$ and 1. The oscillations of such a string can be described by the following wave equation

$$\begin{cases} \partial_t^2 u = \partial_x^2 u, & t > 0, x \in (0, 1) \\ u(t, x) = 0, & t > 0, x = 0, 1 \end{cases}. \tag{2.1}$$

We first separate the variables and suppose $u(t, x) = A(t)B(x)$; this gives us

$$\frac{A''(t)}{A(t)} = \frac{B''(x)}{B(x)} = -\lambda^2,$$

as there would be no oscillations if the right hand side were positive. Inserting the boundary conditions $B(0) = B(1) = 1$ suggests $\lambda = n\pi$ and gives

$$A(t) = a_n \cos n\pi t + b_n \sin n\pi t, \qquad B(x) = \sin n\pi x.$$

Here, we have put the coefficient of B into that of A without loss of generality. Since the equation is linear, any linear combinations of the solutions is again a solution. Also, by symmetry of the sines and cosines, it suffices to consider the case $n \geq 1$. Thus, we get the solution of the form

$$u(t,x) = \sum_{n\geq 1} (a_n \cos n\pi t + b_n \sin n\pi t) \sin n\pi x. \tag{2.2}$$

This suggests that we should think of u as a periodic function with period 2. The motion of the string will be determined if we know the initial position and velocity of every point x, and this is precisely what we need to determined the coefficients a_n and b_n's. Indeed, if we know

$$u(0,x) = f(x), \qquad \partial_t u(0,x) = g(x),$$

then by the orthogonality of the sine's, we have

$$a_n = 2\int_0^1 f(x)\sin(n\pi x)dx, \qquad b_n = \frac{2}{n\pi}\int_0^1 g(x)\sin(n\pi x)dx.$$

Thus, the formula (2.2) with the above expressions of a_n and b_n gives the solution to the wave equation (2.1) with initial position f and initial velocity g.

Now we can define f and g on the interval $[-1,0]$ by odd extension, so they can be viewed as odd periodic functions with period 2. Using the trigonometric identities

$$\cos n\pi t \sin n\pi x = \frac{1}{2}\big(\sin n\pi(x+t) + \sin n\pi(x-t)\big)$$
$$\sin n\pi t \sin n\pi x = \frac{1}{2}\big(\cos n\pi(x-t) - \cos n\pi(x+t)\big)$$

we can in fact express the solution u by

$$u(t,x) = \frac{1}{2}\big(f(x+t) + f(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy.$$

This is *d'Alembert's formula*, which suggests that u can be thought of as the superposition of two waves, one traveling to the left with speed 1, and the other traveling to the right with the same speed. As opposed to the behavior of the solution of heat equation, we can see that the solution to the wave equation has the following properties:

1. It has *finite speed propagation.* If f and g are 0 on some interval, then it will take a positive amount of time for the wave to reach points in the interior of that interval.

2. Unlike the irreversibility of the heat flow, the solution to the wave equation could be reversed in time.

We will go back to these points in more detail when we come to these equations on $\mathbb{R}^d$, and we will see that the behavior of the wave equation will also be very different in even and odd dimensions!