Pointwise convergence of Fourier series:
the theorems of Fejér and Dirichlet

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and $f : \mathbb{T} \to \mathbb{C}$. The n-th Fourier coefficient of f for $n \in \mathbb{Z}$ is defined by

$$a_n = \int_0^1 f(x)e^{-2\pi inx}dx.$$

The Fourier series of f is thus given by

$$f(x) \sim \sum_n a_n e^{2\pi inx}.$$

We use '$\sim$' rather than the equality sign as the convergence of the above series is still not clear. Indeed, one could ideally hope the series to converge to $f(x)$ pointwise, but it simply turns out not to be true. In fact, we have the following negative result.

Proposition 1. *There exists a continuous function on* $[0, 1]$ *whose Fourier series diverges at* 0.

The existence of such a function can be shown either in an abstract way via the uniform boundedness principle, or by explicit construction. We will give an outline for both in the problem sets.

Since the Fourier series of a continuous function f may fail to converge at a given point x, we need to interpret the sum in a different way. For a sequence of real numbers $a_1, a_2, \cdots$, the partial sum s_n is given by

$$s_n = a_0 + \cdots a_n.$$

The *Cesàro sum* of $\{a_k\}$, or the *Cesàro mean* of $\{s_k\}$, is defined by

$$\sigma_n = \frac{1}{n}(s_1 + \cdots s_{n-1}).$$

It is easy to see there are situations where $\{\sigma_n\}$ converges while $\{s_n\}$ fails to converge. We now explore the convergence properties of Fourier series in the sense of Cesàro. For each $n \geq 0$, we define the n-th partial sum of the Fourier series of f by

$$(S_n f)(x) := \sum_{|k|\leq n} a_k e^{2\pi ikx} = \int_0^1 f(y)D_n(x-y)dy,$$

where

$$D_n(x) = \sum_{|k|\leq n} e^{2\pi ikx} = \frac{\sin\big((2n+1)\pi x\big)}{\sin(\pi x)} \tag{1}$$

is the n-th *Dirichlet kernel*. We have already seen that $(S_n f)(x)$ may not converge to $f(x)$. A compromise to get the convergence is to consider the *Cesàro* mean of $(S_n f)(x)$, which we denote by $(\sigma_n f)(x)$. Indeed, we have

$$(\sigma_n f)(x) = \frac{1}{n}\sum_{k=0}^{n-1}(S_k f)(x) = \int_0^1 f(y)F_n(x-y)dy,$$

where

$$F_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} D_k(x) = \frac{1}{n} \left(\frac{\sin(n\pi x)}{\sin(\pi x)} \right)^2. \tag{2}$$

is the n-th Fejér kernel.

Exercise 2. *Verify the expressions of Dirichlet and Fejér kernels in* (1) *and* (2).

Remark 3. Many existing textbooks use different expressions for these kernels, up to a rescale and multiple of 2π. This is because they consider the Fourier series of functions of periodicity 2π, while we assume our functions to have periodicity 1 in order to unify the notations in our course.

We now come back to the convergence properties of $(S_n f)(x)$. In this case, it is more convenient to consider the interval $[-\frac{1}{2}, \frac{1}{2}]$ instead of $[0, 1]$. Of course, nothing will be changed as all functions/kernels involved have periodicity 1. It is easy to verify that the kernels F_n's, now defined on $[-\frac{1}{2}, \frac{1}{2}]$, satisfy

1. $\int_{-\frac{1}{2}}^{\frac{1}{2}} F_n(x) dx = 1$;
2. For any $\delta > 0$, we have $\int_{\delta < |x| \leq \frac{1}{2}} |F_n(x)| dx \to 0$ as $n \to +\infty$.

The first property is an immediate consequence of $\int_{-\frac{1}{2}}^{\frac{1}{2}} D_k(x) dx = 1$, and the second follows from the fact that $\sin(\pi x)$ is bounded away from 0 when $|x| > \delta$. In addition, the kernels F_n are all positive. These properties of F_n gives the following Fejér's theorem.

Theorem 4 (**Fejér**). *Let f be continuous and periodic on $[-\frac{1}{2}, \frac{1}{2}]$. Then the Cesàro mean of $(S_n f)(x)$ converges to $f(x)$ for every x.*

Proof. Since $\int_{-\frac{1}{2}}^{\frac{1}{2}} F_n(y) dy = 1$, for any $\delta > 0$, we have

$$\begin{aligned} (\sigma_n f)(x) - f(x) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \big(f(x-y) - f(x) \big) F_n(y) dy \\ &= \int_{|y| < \delta} \big(f(x-y) - f(x) \big) F_n(y) dy + \int_{\delta < |y| < \frac{1}{2}} \big(f(x-y) - f(x) \big) F_n(y) dy. \end{aligned}$$

Now fix arbitrary $\epsilon > 0$, and choose δ such that $|f(x-y) - f(x)| < \epsilon$ whenever $|y| \leq \delta$. Thus, the first integral is bounded by ϵ uniformly in n, while the second is also smaller than ϵ for all large n. □

We have the following immediate corollary.

Corollary 5. *If two continuous periodic functions have the same Fourier series, then they are equal.*

Also, a closer look at the proof of Fejér's theorem above suggests that the convergence is actually uniform in the sense that

$$\sup_x |(\sigma_n f)(x) - f(x)| \to 0.$$

In fact, since continuous functions on compact sets are automatically uniformly continuous, δ can be chosen independent of the point x, so the uniform convergence follows immediately. Since the sum $S_n f$ is a trigonometric polynomial for each n, so is the Cesàro mean σ_n, we thus have the following very important proposition.

Proposition 6. *Continuous functions on the circle can be uniformly approximated by trigonometric polynomials.*

On the other hand, the above proof would not work if we replace F_n by D_n, as we have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |D_n(x)| dx \sim \log n.$$

This can be easily seen from the fact that the integral of $D_n(x)$ over the k-th peak of the oscillating sine is approximately $\frac{1}{k}$. However, a closer look at the problem reveals that the second integral still vanishes as $n \to +\infty$ thanks to the Riemann-Lebesgue lemma, while the first one remains problematic. In fact, as we have seen before, there exists a continuous f such that $(S_n f)(x)$ does not converge at point x. We thus need to seek further conditions on f that guarantees the desired pointwise convergence. This is the content of Dirichlet's theorem. We first give the Riemann-Lebesgue lemma, which we will be using in the proof, as well as in many occasions later on.

Proposition 7 (**Riemann-Lebesgue**). *Let $f \in L^1([a, b])$. Then we have*

$$\hat{f}(\xi) := \int_a^b f(x) e^{-2\pi i \xi x} dx \to 0$$

as $|\xi| \to +\infty$.

Proof. The claim is true for indicator function of a sub-interval of $[a, b]$, so it is for linear combinations of such functions. Since step functions are dense in L^1, we can choose a sequence of step functions $\{g_k\}$ such that $\|f - g_k\|_{L^1} \to 0$. Now, we write

$$\hat{f}(\xi) = \int_a^b \big(f(x) - g_k(x)\big) e^{-2\pi i \xi x} dx + \int_a^b g_k(x) e^{-2\pi i \xi x} dx.$$

Fix $\epsilon > 0$, choose k large enough such that $\|f - g_k\|_{L^1} < \frac{\epsilon}{2}$, so that the first integral is bounded by $\frac{\epsilon}{2}$. Since g_k is a step function, we can choose ξ large enough so that the second integral is also bounded by $\frac{\epsilon}{2}$. Thus, $|\hat{f}(\xi)|$ can be made smaller than ϵ as long as $|\xi|$ is large enough. Note that the proof is still valid if we choose $a = -\infty$ and $b = +\infty$. $\square$

We now investigate what we can say about $S_n f = f * D_n$. Again, we write

$$\begin{aligned}(S_n f)(x) - f(x) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \big(f(x-y) - f(x)\big) D_n(y) dy \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{f(x-y) - f(y)}{\sin(\pi y)} \sin\big((2n+1)\pi y\big) dy.\end{aligned}$$

If both $f(y)$ and $h_x(y) = \frac{f(x-y)-f(y)}{\sin(\pi y)}$ are integrable, then Riemann-Lebesgue lemma will give the convergence of $(S_n f)(x)$ to $f(x)$. Furthermore, given that $f \in L^1$, the integrability of h_x is the same as the function obtained by replacing the denominator $\sin(\pi y)$ with y, as they have the same singularity near 0. We have thus established Dirichlet's theorem.

Theorem 8 (**Dirichlet**). *If both $f(y)$ and $h_x(y) = \big(f(x-y) - f(x)\big)/y$ are integrable on $[-\frac{1}{2}, \frac{1}{2}]$, then $(S_n f)(x) \to f(x)$.*

It is very important to note that although we have both

$$\int_{\delta < |y| \leq \frac{1}{2}} f(y) F_n(y) dy \to 0, \qquad \text{and} \qquad \int_{\delta < |y| \leq \frac{1}{2}} f(y) D_n(y) dy \to 0,$$

the reasons they vanish as $n \to +\infty$ are *different*. The first integral goes to 0 because $|F_n|$ is small outside the origin, while the second one vanishes because of *cancellations* (by Riemann-Lebesgue)! In particular, it would not vanish if we replace D_n by $|D_n|$.

Dirichlet in 1829 proved a weaker version of the above theorem: a piecewise smooth function converges at every point to the average of its left and right limits. The decisive result about the pointwise convergence of Fourier series was obtained by Carleson in 1966, who proved that the Fourier series of any L^2 function converges almost everywhere. Later, Hunt (1968) extended this result to L^p functions for all $p > 1$. The proofs of Carleson and Hunt are very difficult and far beyond the scope of this course, so we will not discuss it here.

Finally, we should note that the pointwise convergence statement is *not true* if we only assume $f \in L^1$. Kolmogorov (1926) constructed an integrable function whose Fourier series diverges at every point!