Problem set 1

MA433, Autumn term 2014

Questions with a * are good exercises to practice your estimation skills.

Problem 1. Verify the following statements, which we will be frequently using throughout the course. All functions concerned are defined on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. In other words, they are periodic with periodicity 1.

(a) $\int_0^1 f(x)dx = \int_a^{a+1} f(x)dx$ for any a.

(b) $\int_0^1 f(x-y)g(y)dy = \int_0^1 f(y)g(x-y)dy$, so we defined the convolution $(f * g)(x)$ as either of them.

(c) If the n-th Fourier coefficients of f and g are a_n and b_n, then show that the n-th Fourier coefficient of $f * g$ is $a_n b_n$.

Problem 2. Recall that the Fejér kernel is defined by

$$F_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k(x) = \frac{1}{n \sin(\pi x)} \sum_{k=0}^{n-1} \sin\big((2k+1)\pi x\big).$$

Compute the sine series and verify that $F_n(x) = \frac{1}{n}\big(\frac{\sin(n\pi x)}{\sin(\pi x)}\big)^2$.

Problem 3. Evaluate the integral $\int_0^{+\infty} \frac{\sin x}{x} dx$. (Hint: think about $\int_{-\frac{1}{2}}^{\frac{1}{2}} D_n(x)dx$. What goes wrong if you consider $\int_0^1 D_n(x)dx$ instead?)

Problem 4. Let f be a piecewise continuous function on $[-\frac{1}{2}, \frac{1}{2}]$, and let x be a jump discontinuity of f. Modify the argument in class to show that the Cesàro mean of $(S_n f)(x)$ converges to $\frac{1}{2}\big(f(x-) + f(x+)\big)$, where $f(x-)$ and $f(x+)$ are the left and right limits of f at x. Similarly, show that if f is smooth everywhere except having a jump discontinuity at x, then $(S_n f)(x)$ converges to $\frac{1}{2}\big(f(x-) + f(x+)\big)$.

Problem 5. Let $f(x) = |x|$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$.

(a) Compute the Fourier coefficients of f.

(b) Evaluate the sums

$$\sum_{n=1}^{+\infty} \frac{1}{(2n-1)^2} \quad \text{and} \quad \sum_{n=1}^{+\infty} \frac{1}{n^2}.$$

(c) Evaluate the sums

$$\sum_{n=1}^{+\infty} \frac{1}{(2n-1)^4} \quad \text{and} \quad \sum_{n=1}^{+\infty} \frac{1}{n^4}.$$

Problem 6. This problem is about how the decay of Fourier coefficients of a function is related to its regularity. For $\alpha \in (0,1)$, we say $f \in \mathcal{C}^\alpha(\mathbb{T})$ (or α-Hölder continuous) if there exists $C > 0$ such that

$$|f(x+h) - f(x)| < C|h|^\alpha \tag{1}$$

for all x and h. For $\alpha > 1$ but not integer, we say $f \in \mathcal{C}^\alpha(\mathbb{T})$ if f is $\lfloor \alpha \rfloor$ times differentiable, and its $\lfloor \alpha \rfloor$-th derivative belongs to $\mathcal{C}^{\alpha - \lfloor \alpha \rfloor}(\mathbb{T})$. The situation for α being an integer is slightly different: we say $f \in \mathcal{C}^k$ if f has k continuous derivatives. For example, $\mathcal{C}^1$ means continuously differentiable, while the condition (1) with $\alpha = 1$ means Lipschitz.

(a) Show that if $f \in \mathcal{C}^\alpha(\mathbb{T})$ for $\alpha \in (0,1)$, then its n-th Fourier coefficient has decay $a_n = \mathcal{O}(|n|^{-\alpha})$ (that is, $\sup_n |n|^\alpha |a_n| < +\infty$).

(b) Deduce that the conclusion of part (a) is true for every $\alpha > 1$.

(c) In the case of integers, we can say more. Show that if $f \in \mathcal{C}^k$, then $a_n = o(|n|^{-k})$ (that is, $|n|^k a_n \to 0$).

(d*) For α not being an integer, the decay rate cannot be improved. For example, let $\alpha \in (0,1)$ and consider the function

$$f(x) = \sum_{k=0}^{+\infty} 2^{-k\alpha} e^{2\pi i 2^k x}.$$

Then $f \in \mathcal{C}^\alpha$ but $a_n = n^{-\alpha}$ whenever $n = 2^k$, so $|n|^\alpha |a_n|$ does not converge to 0.

Problem 7. This problem investigates some partial converse result to the previous one.

(a) Let a_n be the n-th Fourier coefficient of f. Show that, if

$$\sum_{n \in \mathbb{Z}} |n a_n| < +\infty,$$

then f is continuously differentiable. Deduce that more generally, if

$$\sum_{n \in \mathbb{Z}} |n|^\ell |a_n| < +\infty,$$

then $f \in \mathcal{C}^\ell(\mathbb{T})$. In particular, this implies that if $a_n = \mathcal{O}(|n|^{-\alpha})$ with $\alpha > \ell + 1$, then f is ℓ times continuously differentiable.

(b) We could weaken the assumption if we do not require the derivative to be continuous. Show that if $a_n = \mathcal{O}(|n|^{-\alpha})$ where $\alpha > \frac{3}{2}$, then f is differentiable and $f' \in L^2(\mathbb{T})$. More generally, for any integer $\ell < \alpha - \frac{1}{2}$, the ℓ-th derivative $f^{(\ell)}$ exists, and belongs to $L^2(\mathbb{T})$.

Problem 8. This problem outlines a proof of the existence of a continuous function whose Fourier series diverges at a given point. We make use of the uniform boundedness principle, which we state below.

Let $\mathcal{X}$ be a Banach space, and $\mathcal{Y}$ be a normed vector space. Suppose $\mathcal{T}$ is a collection of continuous linear operators from $\mathcal{X}$ to $\mathcal{Y}$. If for all $f \in \mathcal{X}$ we have

$$\sup_{T \in \mathcal{T}} \|T(f)\|_{\mathcal{Y}} < +\infty,$$

then we have

$$\sup_{T \in \mathcal{T}} \|T\| < +\infty,$$

where $\|T\|$ is defined as $\sup_{\|f\|_{\mathcal{X}}=1} \|T(f)\|_{\mathcal{Y}}$.

Now fix $x \in \mathbb{T}$, and we want to give the existence of a continuous function f such that $(S_n f)(x) \to +\infty$.

(a) Let $\mathcal{X} = \mathcal{C}(\mathbb{T})$, the space of continuous functions on $\mathbb{T}$ with uniform norm, and $\mathcal{Y} = \mathbb{R}$. For any n, let $T_{n,x}$ denote the map $\mathcal{C}(\mathbb{T}) \mapsto \mathbb{R}$ such that

$$T_{n,x}(f) = (S_n f)(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x-y) D_n(y) dy.$$

Convince yourself that $T_{n,x}$ is a linear.

(b) The norm on $\mathcal{C}(\mathbb{T})$ is the uniform norm defined by $\|f\| = \sup_{x \in \mathbb{T}} |f(x)|$. Why do we have

$$\|T_{n,x}\| := \sup_{\|f\|=1} |T_{n,x}(f)| = \int_{-\frac{1}{2}}^{\frac{1}{2}} |D_n(y)| dy?$$

(A graphic explanation would be sufficient.)

(c*) Let $\mathcal{T}$ denote the collection of operators $\{T_{n,x}\}$. Show that $\int_{-\frac{1}{2}}^{\frac{1}{2}} |D_n(y)| dy$ grows like $\log n$, so

$$\sup_{T \in \mathcal{T}} \|T\| = +\infty.$$

Conclude that there exists $f \in \mathcal{C}(\mathbb{T})$ such that $S_n f$ diverges at x.