Problem set 2

MA433, Autumn term 2014

1 Some Fourier series

Problem 1. Let α be a non-integer, and consider the function

$$f(x) = \frac{\pi}{\sin \pi\alpha} e^{-2\pi i x\alpha}$$

on $[-\frac{1}{2}, \frac{1}{2}]$. Compute its Fourier series and evaluate the sum

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n+\alpha)^2}.$$

Is it a contradiction to the fast decay of Fourier coefficients of smooth functions?

Problem 2. (a) Let f be piecewise $\mathcal{C}^1$ on $\mathbb{T}$ with $\int_0^1 f(x)dx = 0$. Show that

$$\int_0^1 |f(x)|^2 dx \leq \frac{1}{4\pi^2} \int_0^1 |f'(x)|^2 dx,$$

with equality if and only if $f(x) = A\cos(2\pi x) + B\sin(2\pi x)$.

(b) Let f be a continuously differentiable function on $[a, b]$ with $f(a) = f(b) = 0$. Show that

$$\int_a^b |f(x)|^2 dx \leq \frac{(b-a)^2}{\pi^2} \int_a^b |f'(x)|^2 dx.$$

When does equality hold? (Hint: we would like to use part (a) but f here may not integrate to 0. How can we make the change to the situation to fit that assumption? Also, the interval $[a, b]$ here is never a problem, as one could always make a change of variable to rewrite the claim on any interval.)

(c) The conditions $\int_0^1 f(x)dx = 0$ or f vanishes on the end points of the interval are important here. Find counter examples to the previous two questions when these conditions are not satisfied.

Problem 3 (Bernstein's theorem). Let $f \in \mathcal{C}^\alpha(\mathbb{T})$. We outline a proof below that as long as $\alpha > \frac{1}{2}$, then the Fourier series of f converges absolutely. Note that this is *not* a contradiction to the decay rate $a_n = \mathcal{O}(|n|^{-\alpha})$. The constants C's appearing below may represent different values in different contexts, but all of them are independent of n and h.

(a) For every $h > 0$, we define the function $g_h(x) = f(x+h) - f(x-h)$. Use this function g to show that

$$\sum_{n\in\mathbb{Z}} |\sin(2\pi nh)|^2 |a_n|^2 < Ch^{2\alpha}.$$

(b) In order to extract information of $\sum |a_n|^2$, we need to restrict ourselves to the sum where $|\sin(2\pi nh)|$ is bounded away from 0. Show that, for $h = 2^{-p}$, we have

$$\sum_{2^{p-1}\le|n|<2^p} |a_n|^2 < C2^{-2\alpha p}.$$

(c) Estimate the sum $\sum_{2^{p-1}\le|n|<2^p} |a_n|$, and conclude that $\sum_{n\in\mathbb{Z}} |a_n| < +\infty$ if $\alpha > \frac{1}{2}$.

Remark 1.1. Bernstein's theorem is sharp in the sense that one can find a $\mathcal{C}^{\frac{1}{2}}(\mathbb{T})$ function whose Fourier series does not converge absolutely. The following example, first considered by Hardy-Littlewood, is classic:

$$f(x) = \sum_{n=1}^{+\infty} \frac{e^{in\log n}}{n} e^{2\pi inx}.$$

Interestingly, the above series still converges uniformly on $[0,1]$ with limit in $\mathcal{C}^{\frac{1}{2}}$, despite the fact that the coefficients do not converge absolutely. We do not go into details of the proof here.

Problem 4. Conversely, there is no necessary condition on the regularity of a function for its Fourier series to be absolutely convergent, except the obvious one of being continuous. For example. let $\{\epsilon_n\}$ be any sequence such that $\epsilon_n \to 0$. Find a continuous function f with $\sum_n |a_n| < +\infty$ but $|a_n| > |\epsilon_n|$ for infinitely many n.

In other words, we can find continuous functions with absolutely convergent Fourier series, yet their decay speed can be as slow as we want. Thus, no criterion on the modulus of continuity can be put on f for the absolute convergence of its Fourier series.

2 More on equidistributions

Recall Weyl's criterion states that a sequence of numbers $\{\xi_n\}$ is equidistributed in $[0,1]$ if and only if for any non-zero integer k, the we have

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi ik\xi_n} \to 0 \tag{2.1}$$

as $N \to +\infty$. A closer look at the proof for the equidistribution of $\{n\alpha\}$ suggests that it already implies the 'if' part. The following problem asks you to prove the other part.

Problem 5. Show that if $\{\xi_n\}$ is equidistributed in $[0,1]$, then (2.1) is satisfied for any non-zero integer k. (Hint: it suffices to prove

$$\frac{1}{N}\sum_{n=1}^{N} f(\xi_n) \to \int_0^1 f(x)dx$$

for smooth functions. Start with easy ones as usual...)

Problem 6. Use Weyl's criterion to show that, for any $\alpha \neq 0$ and any real number $\beta \in (0,1)$, the fractional parts of $\{\alpha n^\beta\}$ is equidistributed in $[0,1]$. (Hint: compare $e^{2\pi i \alpha n^\beta}$ with the integral $\int_n^{n+1} e^{2\pi i \alpha x^\beta} dx$.)

Problem 7. Show that $\{\alpha \log n\}$ is *not* equidistributed for any α.

In the original theorem of Weyl, he showed $\{n^2\alpha\}$ is equidistributed if and only α is irrational. Just as usual, the case when α is rational is straightforward, but the other part is much harder than the one considered in class. Since α is irrational, $k\alpha$ is also irrational for any k, thus it suffices to check the sum $\sum_{n=1}^{N} e^{2\pi i \xi_n}$ only. But this time, we have $\xi_n = n^2\alpha$, and the quadratic exponentials are not easy to sum. The trick here is that, for any $1 \leq K \leq N$, we can write

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i \xi_n} = \frac{1}{NK}\sum_{n=1}^{N-K} (e^{2\pi i \xi_n} + \cdots + e^{2\pi i \xi_{n+K}}) + \text{Remainder}. \tag{2.2}$$

We now give an outline of the proof in the following problem.

Problem 8. (a) Check the norm of the remainder in (2.2) is bounded by $\frac{K+1}{N}$. Thus, in considering the limit $N \to +\infty$, we could simply ignore this error term.

(b) Show that there exists a $C > 0$ independent of N and K such that

$$\left|\sum_{n=1}^{N-K} \left(e^{2\pi i \xi_n} + \cdots e^{2\pi i \xi_{n+K}}\right)\right|^2 < CN \sum_{k=0}^{K} \left|\sum_{n=1}^{N-K} e^{2\pi i (\xi_{n+k} - \xi_n)}\right|.$$

(c) Use the above estimate and (2.2) to prove $\{n^2\alpha\}$ is equidistributed in $[0,1]$ if α is irrational.