Problem set 3

MA433, Autumn term 2014

Problem 1. Prove that if a function and its Fourier transform both have compact support, then it must be identically 0.

Problem 2. Find a continuous function $f \in L^1(\mathbb{R})$ such that its Fourier transform does not belong to $L^1(\mathbb{R})$.

Problem 3. Show that, for all $f, g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, we have

$$\|f * g\|_2^2 \leq \|f * f\|_2 \|g * g\|_2.$$

Is such an inequality possible if we replace all the L^2 norms above by L^1?

Problem 4. Let $H_t(x)$ be the heat kernel on $\mathbb{R}^d$. Show that it solves the heat equation

$$\partial_t H = \Delta H$$

with initial condition δ.

Problem 5. Find the solution to the inhomogeneous heat equation

$$\partial_t u = \Delta u + g(t, x)$$

on $\mathbb{R}^d$, with initial data $u(0, x) = f(x)$. You can assume f and g are both smooth functions with rapid decay. Explain the solution in terms of Duhamel's principle.

Problem 6. This is an analogue to the statement that the Cesàro mean of the Fourier series of a continuous function f on the circle converges uniformly to f. To make things simple, we work on the real line $\mathbb{R}$. Suppose $f \in L^1(\mathbb{R})$, and consider the Cesàro sum

$$(\sigma_N f)(x) = \int_{-N}^{N} (1 - \frac{|\xi|}{N}) \hat{f}(\xi) e^{2\pi i \xi x} d\xi.$$

(a) Compare the above integral with the definition of $\sigma_N f$ in the case of Fourier series. Explain why it is called Cesàro sum. Note that N is not required to be an integer here.
(b) Find the expression of Fejér's kernel on the real line (F_N) such that $\sigma_N f = f * F_N$. Show that

$$\int_{\mathbb{R}} F_N(x) dx = 1.$$

(c) Show that $(\sigma_N f)(x) \to x$ whenever f is continuous at x.

Problem 7. Recall the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

for nice functions f. The purpose of this problem is to show that the above formula can fail if f does not decay fast enough, even if both sides above converge absolutely.

(a) Let g be a smooth function (on $\mathbb{R}$) with compact support in the interval $[0,1]$. For each integer N, let

$$g_N(x) = \frac{1}{N} \sum_{|k|\leq N-1} (1 - \frac{|k|}{N}) g(x+k).$$

Show that $\hat{g}_N(\xi) = \frac{1}{N} F_N(\xi)\hat{g}(\xi)$, where F_N is the Fejér's kernel on the circle and periodically extended to the whole real line.

(b) Take for granted that there exists smooth non-negative functions $\{g^{(j)}\}$ such that each $\widehat{g^{(j)}} \in L^1(\mathbb{R})$ and

$$\sum_{j=1}^{+\infty} g^{(j)}(x) = \begin{cases} 1, x \in (0,1) \\ 0, x \notin (0,1) \end{cases}.$$

(Actually the above condition plus the smoothness guarantees each of the Fourier transforms will be in L^1.) Each of the $g^{(j)}$'s satisfies the assumptions in part(a), so we choose for each j an integer N_j and get a function $g^{(j)}_{N_j}$. Now, define

$$f(x) := \sum_{j=1}^{+\infty} g^{(j)}_{N_j}(x).$$

Show that we can choose $N_j \to +\infty$ fast enough such that the function f fails to satisfy the Poisson summation formula. What are $f(n)$'s and $\hat{f}(n)$'s then?