Problem set 4

MA433, Autumn term 2014

Problem 1. Let T be a tempered distribution in $d \geq 3$ dimensions, whose Fourier transform $\hat{T}$ is given by the function $-\frac{1}{4\pi^2|\xi|^2}$. Without computing the form of T, verify directly that $\Delta T = \delta$.

Problem 2. Let T be a distribution, and ψ be a test function. For every $x \in \mathbb{R}^d$, define the function $\tilde{\psi}_x$ by

$$\tilde{\psi}_x(y) := \psi(x-y).$$

The convolution $T * \psi$ can be defined either as a function by

$$(T * \psi)(x) = T(\tilde{\psi}_x),$$

or as a distribution by

$$(T * \psi)(\varphi) = T(\tilde{\psi}_0 * \varphi).$$

Show that these two definitions agree, and $T * \psi$ is a smooth function.

Problem 3. Let T be a tempered distribution. Show that $D^\alpha(T * \varphi) = D^\alpha T * \varphi = T * D^\alpha \varphi$.

Problem 4. Let $f \in L^2(\mathbb{R}^d)$ and α be a multi-index. Show that there exists $g \in L^2(\mathbb{R}^d)$ such that $g = D^\alpha f$ in the distributional sense[1] if and only if

$$(2\pi i \xi)^\alpha \hat{f}(\xi) \in L^2(\mathbb{R}^d),$$

and in this case, it is just the Fourier transform of g.

Remark 0.1. Note that the weak derivative of f being a function does not necessarily mean f is differentiable, or even continuous! In general, to infer certain regularity of f, one needs sufficiently many weak derivatives (as functions) and sufficient integrability of them. Statements of this kind are generally called Sobolev embedding theorems. One example is the next problem.

[1] This means

$$\int g\varphi dx = (-1)^{|\alpha|} \int f D^\alpha \varphi dx$$

for every test function φ.

Problem 5. Let $f \in L^2(\mathbb{R}^d)$, and n be the smallest integer that is strictly bigger than $\frac{d}{2}$. Suppose for all $1 \leq |\alpha| \leq n$, the weak derivative $D^\alpha f$ exists and belongs to $L^2(\mathbb{R}^d)$. Show that f then can be modified on a set of measure 0 such that it is continuous and bounded. (Hint: try to show $\hat{f} \in L^1$.) In fact, we can say more about f than just being continuous, but that requires deeper work, and we do not attempt to do it here.

Problem 6. The conclusion of the previous problem is false when $n = \frac{d}{2}$. Consider the case $d = 2$, and let

$$f(x) = \big(\log(1/|x|)\big)^\alpha \eta(x),$$

where $\alpha \in (0, \frac{1}{2})$, and η is a smooth cutoff function with $\eta(x) = 1$ for $|x| \leq \frac{1}{10}$ and $\eta(x) = 0$ for $|x| \geq \frac{1}{2}$. Show that both weak derivatives $\frac{\partial}{\partial x_1} f$ and $\frac{\partial}{\partial x_2} f$ are $L^2(\mathbb{R}^2)$ functions, but f cannot be modified on a set of measure 0 such that it becomes continuous at the origin.

Problem 7 (Nash's inequality). The following inequality, now known as Nash's inequality, was first introduced in John Nash's seminal paper in 1958, titled 'Continuity of solutions of parabolic and elliptic equations'. It states that, there exists a constant C depending on dimension d only such that for every $f \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\|f\|_2 < C\|f\|_1^\alpha \|\nabla f\|_2^\beta \tag{0.1}$$

for some positive exponents α and β. Here, the important fact is that C is *independent* of f, and so are α and β. Nash acknowledged Elias Stein for giving a quick proof, which is outlined below.

(a) What is the *only* possible pair (α, β) for such an inequality to be true?

(b) Show that, there exists $C_1, C_2 > 0$ such that

$$\int_{\mathbb{R}^d} |f(x)|^2 dx < C_1 \lambda^d \|f\|_1^2 + \frac{C_2}{\lambda^2} \|\nabla f\|_2^2$$

for any $\lambda > 0$ and $f \in \mathcal{S}(\mathbb{R}^d)$.

(c) Prove Nash's inequality (0.1) for the correct values (α, β).