Problem set 5

MA433, Autumn term 2014

Problem 1. Let f be a continuous function on $\mathbb{R}$ with compact support (and hence bounded). If $\int f(x)dx = a$, show that

$$(Hf)(x) = \frac{a}{\pi x} + \mathcal{O}(\frac{1}{x^2})$$

for all large x.

Problem 2. Let f be a positive continuous function with compact support. Let $0 < \alpha < d$, and let

$$(Tf)(x) = \int_{\mathbb{R}^d} |x-y|^{-\alpha} f(y)dy.$$

Show that there exists $c, C > 0$ such that

$$c|x|^{-\alpha} < (Tf)(x) < C|x|^{-\alpha}$$

for all large $|x|$. In this case, the decay rate of Tf is the same as that of the kernel.

Problem 3. Let $\alpha, \beta \in (0, d)$ with $\alpha + \beta > d$, and $G = (|\cdot| + \epsilon)^{-\alpha} * |\cdot|^{-\beta}$. More precisely, we have

$$G(x) = \int_{\mathbb{R}^d} (|y| + \epsilon)^{-\alpha} |x-y|^{-\beta} dy.$$

Show that there exists $c, C > 0$ such that

$$c(|x| + \epsilon)^{-\alpha-\beta+d} < G(x) < C(|x| + \epsilon)^{-\alpha-\beta+d}.$$

for all x. Note that the addition of ϵ is just to make G to be defined at 0. This problem says the convolution of these two kernels has a better singularity (than both of the kernels) at the origin, while it has a slower decay at infinity (remember the condition that both $\alpha, \beta < d$ and $\alpha + \beta > d$), and the degree of the singularity as well as the decay are quantified.

Problem 4. Let $f \in L^p$ and $p_1 < p < p_2$ be arbitrary. If we set

$$f_1 = 1_{\{|f| \geq 1\}}, \qquad f_2 = 1_{\{|f| < 1\}},$$

then show that $f_i \in L^{p_i}$. Note that the choice of 1 is arbitrary; the conclusion holds for any cutoff. What is more important is that f_1 contains the singularity of f, and f_2 contains the decay (the tail) of f.

Problem 5. Justify the formal calculations made in class that $\widehat{\mathrm{pv}(1/x)}(\xi) = -\pi i \cdot \mathrm{sgn}(\xi)$.