Cesàro and Abel summability

Consider a sequence $\{a_n\}_{n\geq 0}$, and its partial sum

$$s_n = a_0 + \cdots + a_n.$$

Many times the sequence $\{s_n\}$ does not converge, so we need to interpret the sum in a different way. Usually we can achieve this by looking at weighted averages of the s_n's. For example, the *Cesàro sum* of $\{a_n\}$ is defined by

$$\sigma_n = \frac{1}{n}(s_0 + \cdots + s_{n-1}).$$

It is clear that σ_n is just the arithmetic average of the first n terms in the sequence $\{s_k\}$. There is also a different but more general way to put weights on the s_n's, namely the Abel sum. For each $r \in [0,1)$, define the *Abel sum* $A(r)$ by

$$A(r) = \sum_{k=0}^{+\infty} a_k r^k.$$

If the limit

$$\lim_{r\to 1} A(r)$$

exists and is finite, we call it the Abel limit of $\{a_n\}$. Using summation by parts (a discrete version of integration by parts), we see that the Abel sum and the usual sum are related by

$$\begin{aligned}\sum_{k=0}^{N} a_k r^k &= a_0 + \sum_{k=1}^{N}(s_k - s_{k-1})r^k \\ &= \sum_{k=0}^{N} s_k r^k - \sum_{k=0}^{N-1} s_k r^{k+1} \\ &= (1-r)\sum_{k=0}^{N} s_k r^k + s_N r^{N+1}.\end{aligned}$$

Under ideal circumstances, we would have $s_N r^{N+1} \to 0$, so by sending $N \to +\infty$, we see that the Abel sum is nothing but a weighted average of the s_k's with weights $(1-r)r^k$. The relations between these three types of convergence can be summarized as follows:

Convergence $\Rightarrow$ **Cesàro convergence** $\Rightarrow$ **Abel convergence**,

and none of these arrows can be reversed. We now give proofs of the two above arrows, starting with the first one.

Suppose $s_k \to s$, then there exists K such that for all $k > K$, we have $|s_k - s| < \epsilon$. Now, we have

$$\sigma_n - s = \frac{1}{n}\sum_{k=0}^{K-1}(s_k - s) + \frac{1}{n}\sum_{k=N}^{n-1}(s_k - s).$$

The second term is bounded by ϵ, while the first term goes to 0 as $n \to +\infty$. To verify the second arrow, we take advantage of the above identity relating the Abel and usual sums to get

$$\begin{aligned}\sum_{k=0}^{N} a_k r^k &= (1-r)\sum_{k=0}^{N} s_k r^k + s_N r^{N+1} \\ &= (1-r)\sum_{k=0}^{N} \big((k+1)\sigma_{k+1} - k\sigma_k\big) r^k + s_N r^{N+1} \\ &= (1-r)^2 \sum_{k=0}^{N} k\sigma_k r^{k-1} + (1-r)(N+1)\sigma_{N+1} r^N + s_N r^{N+1}.\end{aligned}$$

If σ_n converges to a limit and $r < 1$, then both $(N+1)\sigma_{N+1} r^N$ and $s_N r^{N+1}$ will vanish as $N \to +\infty$, which immediately gives

$$\sum_{k=0}^{+\infty} a_k r^k = (1-r)^2 \sum_{k=0}^{+\infty} k\sigma_k r^{k-1}$$

for any $r \in (0,1)$. Now we know $\sigma_k \to s$, and we want to show that the right hand side above also converges to s as $r \to 1$. For this, we use the identity (for $r < 1$)

$$\sum_{k=1}^{+\infty} k r^{k-1} = \frac{1}{(1-r)^2},$$

so we have

$$(1-r)^2 \sum_{k=0}^{+\infty} k\sigma_k r^{k-1} - \sigma = (1-r)^2 \sum_{k=1}^{N-1} k r^{k-1}(\sigma_k - s) + (1-r)^2 \sum_{k=N}^{+\infty} k r^{k-1}(\sigma_k - s),$$

where N is an integer such that $|\sigma_k - s| < \epsilon$ for all $k \geq N$. As before, the second term is smaller than ϵ, and the first term vanishes as $r \to 1$. This completes the proof.

For examples illustrating that neither of the two arrows could be reversed, we note that the series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

does not converge to a limit, but it has Cesàro sum $\frac{1}{2}$. Also, the Abel sum of the series

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots$$

is $\frac{1}{4}$, but it is not Cesàro summable. In 1897, Tauber gave a sufficient condition under which Abel summability implies the usual stronger convergence, now known as the (original) Tauberian theorem. We state and give a proof below.

Theorem 1 (Tauber, 1897). *Let a_k be a sequence of complex numbers, assume and*

$$A(r) = \sum_{k=0}^{+\infty} a_k r^k$$

converges for every $r \in (0,1)$. If $A(r) \to A$ as $r \uparrow 1$, and $a_k = o(\frac{1}{k})$ (that is, $ka_k \to 0$), then we have

$$\sum_{k=0}^{+\infty} a_k = A.$$

Proof. Since $\sum a_k r^k$ converges to A when $r \uparrow 1$ along the sequence $r_N = 1 - \frac{1}{N}$, it suffices to show that

$$\sum_{k=0}^{N} a_k - \sum_{k=0}^{+\infty} a_k r^k \to 0$$

along the sequence $r = 1 - \frac{1}{N}$ as $N \to +\infty$. Fix $\epsilon > 0$, we choose K such that $|ka_k| > \epsilon$ for all $k > K$. We then write

$$\begin{aligned}\sum_{k=0}^{N} a_k - \sum_{k=0}^{+\infty} a_k r^k &= \sum_{k=0}^{K} a_k(1-r^k) + \sum_{k=K+1}^{N} a_k(1-r^k) - \sum_{k=N+1}^{+\infty} a_k r^k \\ &= I_1 + I_2 + I_3.\end{aligned}$$

That $I_1 \to 0$ as $N \to +\infty$ is obvious. For I_2, we have

$$\begin{aligned}|I_2| &\leq (1-r) \sum_{k=K+1}^{N} |a_k|(1 + \cdots + r^{k-1}) \\ &\leq (1-r) \sum_{k=K+1}^{N} |ka_k| \\ &\leq (1-r)N\epsilon \leq \epsilon,\end{aligned}$$

where we have used $r = 1 - \frac{1}{N}$. For the third one, we have

$$\begin{aligned}|I_3| &\leq \sum_{k=N+1}^{+\infty} |ka_k| \frac{r^k}{k} \\ &< \frac{\epsilon}{N+1} \sum_{k=0}^{+\infty} r^k \\ &< \epsilon,\end{aligned}$$

again having used the assumption that $1 - r = \frac{1}{N}$. □

Littlewood (1911) weakened the assumption on a_n from being $o(\frac{1}{n})$ to $\mathcal{O}(\frac{1}{n})$. Nowadays, Tauberian theorems in general refers to the statements going from Abel summability to the real summability of a series. Finally, we should make a remark that in the trivial case of positive series with $a_k \geq 0$, all these three convergences are the same. In fact, s_n will be increasing and must have a limit (possibly $+\infty$), and hence the Cesàro and Abel sums will have the same limit.