The wave equation

Consider the wave equation on $\mathbb{R}^+ \times \mathbb{R}^d$ given by

$$\begin{cases} \partial_t^2 u = \Delta u \\ u|_{t=0} = f(x) \\ \partial_t u|_{t=0} = g(x) \end{cases}. \tag{0.1}$$

We assume for simplicity f and g belong to $\mathcal{S}(\mathbb{R}^d)$. As before, we take the Fourier transform of u in space variables. Solving the resulting ODE and plugging in the initial data f and g, we get

$$\hat{u}(t,\xi) = \hat{f}(\xi)\cos(2\pi|\xi|t) + \frac{\hat{g}(\xi)}{2\pi|\xi|}\sin(2\pi|\xi|t).$$

Then, taking the inverse transform gives back the solution u by

$$u(t,x) = \int_{\mathbb{R}^d} \left(\hat{f}(\xi)\cos(2\pi|\xi|t) + \hat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|} \right) e^{2\pi i\xi\cdot x} d\xi.$$

Under the assumptions $f, g \in \mathcal{S}(\mathbb{R}^d)$, by differentiating under the integral sign, it is easy to verify that u is indeed the solution to (0.1) with the correct initial conditions.

However, the Fourier representation above does not tell us much information about the solution. If we are in one dimension, then we have the d'Alembert's formula

$$u(t,x) = \frac{1}{2}\big(f(x+t) + f(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy,$$

which follows directly from the observations

$$\cos(2\pi|\xi|t) = \frac{1}{2}\big(e^{2\pi i|\xi|t} + e^{-2\pi i|\xi|t}\big), \qquad \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|} = \frac{1}{4\pi i|\xi|}\big(e^{2\pi i|\xi|t} - e^{-2\pi i|\xi|t}\big),$$

and that both are even functions in ξ. The d'Alembert's formula tells us the following about the wave equation in 1D:

1. It has finite propagation propagation speed 1.
2. The solution $u(t,x)$ depends on f only at two boundary points $(x+t)$ and $x-t$.
3. $u(t,x)$ depends on g in the interval $(x-t, x+t)$.

Odd dimensions ($d \geq 3$) via spherical means

The wave equation in odd dimensions $d \geq 3$ can be solved by the method of spherical means. For any $x \in \mathbb{R}^d$, define

$$\begin{aligned} U_x(t,r) &:= \fint_{\partial B(x,r)} u(t,y)\sigma_r(dy), \\ F_x(r) &:= \fint_{\partial B(x,r)} f(y)\sigma_r(dy), \\ G_x(r) &:= \fint_{\partial B(x,r)} g(y)\sigma_r(dy), \end{aligned}$$

where $\sigma_r(dy)$ denotes the surface measure on the $d-1$ dimensional sphere with radius r. If we further extend the functions U, F, G into even functions of r on the whole real line, then simple calculations show that, for every x, the extended U solves the Euler-Poisson-Darboux equation

$$\partial_t^2 U = \partial_r^2 U + \frac{d-1}{r}\partial_r U$$

with initial data $U|_{t=0} = F$ and $\partial_t U|_{t=0} = G$. Note that we have omitted x for notational simplicity. This is almost the 1D wave equation except the extra first order derivative of U. To get rid of this extra term, we define for $d = 2k+1$ the new functions $\tilde{U}, \tilde{F}$ and $\tilde{G}$ by

$$\begin{aligned} \tilde{U}_x(t,r) &:= \big(\frac{1}{r}\partial_r\big)^{k-1}\big(r^{2k-1}U_x(t,r)\big), \\ \tilde{F}_x(r) &:= \big(\frac{1}{r}\frac{d}{dr}\big)^{k-1}\big(r^{2k-1}F_x(r)\big), \\ \tilde{G}_x(r) &:= \big(\frac{1}{r}\frac{d}{dr}\big)^{k-1}\big(r^{2k-1}G_x(r)\big). \end{aligned}$$

We have the following useful proposition.

Proposition 1. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth. Then, we have*

$$\frac{d^2}{dr^2}\big(\frac{1}{r}\frac{d}{dr}\big)^{k-1}\big(r^{2k-1}\phi(r)\big) = \big(\frac{1}{r}\frac{d}{dr}\big)^{k}\big(r^{2k}\phi'(r)\big),$$

and

$$\big(\frac{1}{r}\frac{d}{dr}\big)^{k-1}\big(r^{2k-1}\phi(r)\big) = \sum_{j=0}^{k-1}\beta_j^{(k)} r^{j+1}\phi^{(j)}(r),$$

where $\beta_j^{(k)}$'s are fixed numbers depending on j and k only. In particular, we have $\beta_0^{(k)} = (2k-1)!!$.

Using the first identity in the above proposition, one can easily see that the function $\tilde{U}$ defined above satisfies the 1D wave equation

$$\partial_t^2 \tilde{U} = \partial_r^2 \tilde{U}$$

with initial conditions $\tilde{U}|_{t=0} = \tilde{F}$ and $\partial_t \tilde{U}|_{t=0} = \tilde{G}$. We could then use d'Alembert's formula to express $\tilde{U}$ by

$$\tilde{U}_x(t,r) = \frac{1}{2}\big(\tilde{F}_x(r+t) + \tilde{F}_x(r-t)\big) + \frac{1}{2}\int_{r-t}^{r+t} \tilde{G}_x(y)dy.$$

It now remains to recover $u(t,x)$ from $\tilde{U}_x(t,r)$. The second expression in the proposition above suggests

$$u(t,x) = \lim_{r\to 0} \frac{\tilde{U}_x(t,r)}{\beta_0^{(k)} r}.$$

This gives us

$$u(t,x) = \frac{1}{\beta_0^{(k)}} \lim_{r\to 0} \frac{1}{2r}\big(\tilde{F}_x(r+t) + \tilde{F}_x(r-t) + \int_{r-t}^{r+t} \tilde{G}_x(y)dy\big).$$

To simplify this expression, we note that F_x and G_x are even functions in r, so $\tilde{F}_x$ and $\tilde{G}_x$ are both odd in r. We thus have

$$u(t,x) = \frac{1}{\beta_0^{(k)}}\big(\tilde{F}'(t) + \tilde{G}(t)\big).$$

Thus, we obtain the following representation formula for the solution u in $d = 2k+1$ dimensions:

$$u(t,x) = \frac{1}{(2k-1)!!}\Bigg[\partial_t (t^{-1}\partial_t)^{k-1}\bigg(t^{2k-1} \fint_{\partial B(x,t)} f(y)\sigma_t(dy)\bigg) + (t^{-1}\partial_t)^{k-1}\bigg(t^{2k-1} \fint_{\partial B(x,t)} g(y)\sigma_t(dy)\bigg)\Bigg]$$

It is easy to see from the above formula that at any time t, the value $u(t,x)$ depends on the initial data only at space points $\{y : |y-x| = t\}$. Conversely, the initial turbulence at location x only affects the solution at time t at space points $\{y : |y-x| = t\}$. This is the Huygens' principle in odd dimensions $d \geq 3$.

Even dimensions via the method of descent

The solution of the wave equation in even dimensions could be deduced from those in odd dimensions. Suppose u is a solution in dimension $d = 2k$ with initial position f and velocity g, then the function

$$\bar{u}(t,x,x_{2k+1}) := u(t,x)$$

solves (0.1) in dimension $2k+1$ with initial position $\bar{f}(x,x_{2k+1}) := f(x)$ and velocity $\bar{g}(x,x_{2k+1}) = g(x)$. We could then implement the formula for odd dimensions, and get a representation for $d = 2k$ as

$$u(t,x) = \frac{1}{(2k)!!}\Bigg[\partial_t (t^{-1}\partial_t)^{k-1}\bigg(t^{2k} \fint_{B(x,t)} \frac{f(y)}{\sqrt{t^2 - |y-x|^2}} dy\bigg) + (t^{-1}\partial_t)^{k-1}\bigg(t^{2k} \fint_{B(x,t)} \frac{g(y)}{\sqrt{t^2 - |y-x|^2}} dy\bigg)\Bigg].$$

The representation formulas suggest the following:

1. **Huygens principle**: When $d \geq 3$ is odd, the initial disturbance at a point x only influences the locations that are at distance exactly t to x. When d is odd, it has effects on all the locations within distance t to x.

2. Waves propagate at finite speed (in this case 1 here).