Weyl's equidistribution theorem

A celebrated result of Hermann Weyl asserts that, the sequence

$$\alpha, 2\alpha, \cdots, n\alpha, \cdots \qquad \text{mod } 1$$

is uniformly distributed on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ if and only if α is irrational. Here, a sequence $\{\xi_n\}$ is said to be uniformly distributed (or equidistributed) on $[0,1]$ if for any sub-interval $[a,b] \subset [0,1]$, we have

$$\lim_{N\to+\infty} \frac{\#\{\xi_n \in [a,b] : n \leq N\}}{N} = b - a.$$

We first observe that, if α is rational, then the sequence $\{n\alpha\}$ (mod 1) only takes finitely many values, and thus cannot be equidistributed. To prove the other part of Weyl's theorem, we let $\chi_{[a,b]}$ be the function such that

$$\chi_{[a,b)}(x) = \begin{cases} 1, x \in [a,b) \\ 0, x \in [0,1) \setminus [a,b) \end{cases},$$

and periodically extended to the whole real line. Thus, the number of the points in the finite sequence $\{\alpha, 2\alpha, \cdots, N\alpha\}$ could be represented by $\sum_{n=1}^{N} \chi_{[a,b)}(n\alpha)$, and Weyl's theorem can be formulated as follows:

$$\frac{1}{N}\sum_{n=1}^{N} \chi_{[a,b)}(n\alpha) \to b - a$$

if and only if α is irrational. The 'only if' part has been checked above. For the 'if' part, we need the following proposition.

Proposition 1. *Let f be a continuous periodic function on $\mathbb{T}$ and α be irrational. Then we have*

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(x)dx. \tag{1}$$

Before giving a proof of the proposition, we first observe that ideally, one would like to take $f = \chi_{[a,b)}$ to conclude Weyl's theorem, but unfortunately the latter is not continuous on $\mathbb{T}$ and hence does not satisfy the hypothesis of the proposition. On the other hand, for any two continuous functions f and g on $\mathbb{T}$ with $f \leq \chi_{[a,b)} \leq g$, we have

$$\begin{aligned}
\int_0^1 f(x)dx &= \lim_{N\to+\infty} \frac{1}{N}\sum_{n=1}^{N} f(n\alpha) \\
&\leq \liminf_{N\to+\infty} \frac{1}{N}\sum_{n=1}^{N} \chi_{[a,b)}(n\alpha) \\
&\leq \limsup_{N\to+\infty} \frac{1}{N}\sum_{n=1}^{N} \chi_{[a,b)}(n\alpha) \\
&\leq \lim_{N\to+\infty} \frac{1}{N}\sum_{n=1}^{N} g(n\alpha) \\
&= \int_0^1 g(x)dx.
\end{aligned}$$

For the indicator function $\chi_{[a,b)}$, we could choose continuous $f \leq \chi_{[a,b)} \leq g$ such that the difference

$$\int_0^1 \big(f(x) - g(x)\big)dx$$

is arbitrarily small. Thus, it immediately gives

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{n=1}^{N} \chi_{[a,b)}(x) = \int_0^1 \chi_{[a,b)}(x)dx = b - a,$$

and the equidistribution theorem follows.

It now remains to prove Proposition 1. We first take $f(x) = e^{2\pi ikx}$. For $k = 0$, both sides of (1) are 1. For $k \neq 0$, we have

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi ikn\alpha} = \frac{1}{N} \cdot \frac{e^{2\pi ik\alpha}(1 - e^{2\pi ikN\alpha})}{1 - e^{2\pi ik\alpha}} \to 0$$

since the denominator is non-zero thanks to the irrationality of α. Thus, we can deduce that (1) holds for all trigonometric polynomials, which are nothing but linear combinations of the e_k's. Since continuous functions on the circle can be uniformly approximated by trigonometric polynomials, if f is continuous, there exists a trigonometric polynomial P such that $\sup_x |f(x) - P(x)| < \epsilon$. Thus, we have

$$\begin{aligned} &\left|\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) - \int_0^1 f(x)dx\right| \\ \leq &\left|\frac{1}{N}\sum_{n=1}^{N} \big(f(n\alpha) - P(n\alpha)\big)\right| + \left|\frac{1}{N}\sum_{n=1}^{N} P(n\alpha) - \int_0^1 P(x)dx\right| + \left|\int_0^1 \big(P(x) - f(x)\big)dx\right| \end{aligned}$$

Both the first and third term are smaller than ϵ (for all N) by the choice of P, and the second term can also be made smaller than ϵ for all large enough N. This completes the proof of Proposition 1.

The proof above also suggests the following criterion for equidistributed sequences.

Proposition 2 (Weyl's criterion). *A sequence of real numbers $\xi_1, \cdots, \xi_n, \cdots$ in $[0, 1)$ is equidistributed if and only if for all integers $k \neq 0$, we have*

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi ik\xi_n} = 0.$$

The 'if' part has been proved above, and the 'only if' part is left as an exercise.

We end this note with a remark about the sequence $\{\alpha^n\}$ (mod 1). We take $\alpha = \frac{1+\sqrt{5}}{2}$, and ask whether $\{\alpha^n\}$ (mod 1) is equidistributed. It turns out that, for every n, the number

$$\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

is an integer. Since the second term above has absolute value between 0 and 1, the n-th power will make it very close to 0 for all large n. This implies that the fractional part of $\left(\frac{1+\sqrt{5}}{2}\right)^n$ is concentrated near 0 (or 1, as they are equivalent). In particular, it cannot be equidistributed. A characterization of α for the sequence $\{\alpha^n\}$ (mod 1) to be equidistributed is still a very interesting open problem.