Data analysis

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October 8, 2015

Common distributions

- The continuous Uniform$[a, b]$ distribution
- The exponential distribution
- The Normal distribution ($\rightarrow$time series)
- The Binomial distribution
- The Poisson distribution ($\rightarrow$time series)

Transformations of Random Variables

(Sec 2.1) Transformations

- X an r.v.
- $g : \mathbb{R} \to \mathbb{R}$
- $g(X)$ is also a R.V.

Examples:

- Suppose X has p.d.f.

$$f_X(x) = \begin{cases} 1/4 & -1 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

  $g(x) = |x|$ or $g(x) = 1_{x>1}$

- Suppose X has p.d.f.

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

  $g(x) = -\log x$ (PRNGs...)

Transformations of Random Variables

(Example 2.1.9):

- $f_x(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$ Normal N(0,1) distribution
- $g(x) = x^2$. $Y = g(X)$.
- $f_Y(y) = \frac{d}{dy}\mathbb{P}(Y < y) = 2\frac{d}{dy}P(X < \sqrt{y})$
  $= 2\,\mathbb{P}(X < x)|_{x=\sqrt{y}} \times \frac{d}{dy}\sqrt{y} = f_X(\sqrt{y})/\sqrt{y} =$
  $\frac{1}{\sqrt{2\pi}} y^{-1/2} \exp(-y/2)$
- This is χ^2_1: χ^2 distribution with one degree of freedom.

Transformations of Random Variables

Examples

- (Thm 2.1.10) Continuous r.v. X with c.d.f. F_X.
- F_X^{-1} is increasing
- $Y := F_X(x)$ has p.d.f.

$$f_Y(y) = \begin{cases} 1 & x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

- Useful for simulation/Monte Carlo integration

Expectation

(Section 2.2)

- Weighted average - weighted by probability
- $\mathbb{E}[X]$
  - Discrete case $\mathbb{E}[X] = \sum_x x \cdot p_X(x)$
  - Continuous case $\mathbb{E}[X] = \int_x x \cdot f_X(x)x$
- $\mathbb{E}[g(X)]$
  - Discrete case $\mathbb{E}[g(X)] = \sum_x g(x) \cdot p_X(x)$
  - Continuous case $\mathbb{E}[g(X)] = \int_x g(x) \cdot f_X(x)dx$
- Linearity: $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y] \text{ for } a, b \in \mathbb{R}$ (independence?)

Ex 2.2.2 The Exponential distribution

- The exponential rate(λ) distribution is defined by

$$f_X(x) = \begin{cases} \lambda \exp(-\lambda x) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(sometimes this is called the exponential($1/\lambda$) distribution)

- What is the c.d.f.?
- What is the mean?

Ex 2.2.3 The Binomial distribution

A discrete random variable has the $\text{Bin}(n, p)$ distribution if the p.m.f. is

$$f_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, \ldots, n$$

Mean value

- Directly?
- Writing $X = \sum_{i=1}^{n} B_i$ with independent Bernoulli(p)

$$B_i = \begin{cases} 1 & \textit{probability } p \\ 0 & \textit{probability } 1-p \end{cases}$$

  ?

Ex 2.2.4 The Cauchy distribution

- $f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}$
- $\int \frac{1}{1+x^2}dx = \arctan(x) + C$
- Symmetric about x=0?
- Mean of X? Mean of $|X|$?
- $\int \frac{x}{1+x^2}dx = \frac{1}{2}\log(1+x^2) + C$

Properties of expectation

(Thm 2.2.5)

- Linearity: $\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b[Y] + c$
- Positivity: If $\mathbb{P}(X \geq 0) = 1$ then $\mathbb{E}[X] \geq 0$

Also:

- Independent X and Y: $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$

Variance

- The variance of r.v. X is defined by

$$Var(X) = \mathbb{E}[(X - \mathbb{E}X)^2] = \mathbb{E}[X^2] - (\mathbb{E}X)^2$$

- N.B. (Ex 2.2.6)

  $\mathbb{E}X = \arg\min_b \mathbb{E}[(X - b)^2].$

  (Why the second power?)

Characteristic functions

- (Sec 2.6) r.v. X.
- Characteristic function $\phi_X(t) = \mathbb{E}[\exp(itX)]$.
- $\phi_{aX}(t) = \phi_X(at)$
- $\phi_{X+b}(t) = e^{itb}\phi_X(t)$
- Independence X, Y: $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$

- Thm 2.6.1 Convergence:
  Sequence of r.v.s (X_k) such that

$$\lim_{k \to \infty} \phi_{X_k}(t) \;\; \to \;\; \phi_X(t)$$

in a neighborhood of 0.

Then for all x such that F_X is continuous at x,

$$\lim_{k \to \infty} F_{X_k}(x) = F(x)$$

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Poisson approximation

- The Poission distribution
  - $\mathbb{P}[X = k] = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, \ldots$
  - Characteristic function $\phi_X(t) = \exp[\lambda(e^{it} - 1)]$
- Binomial distribution
  - $\mathbb{P}[X = k] = \binom{n}{k} p^k (1 - p)^{n-k}$
  - Characteristic function $\phi_X(t) = (1 - p + pe^{-it})^n$
- Convergence: $\text{Bin}(n, \lambda/n) \to$Poisson$(\lambda)$
  "Law of small numbers"

Weak Law of Large Numbers

- Convergence in distribution:
  $X_i \xrightarrow{D} X$ if for all $x \in \mathbb{R}$ such that F is continuous at x,
  $F_{X_n}(x) \to F_X(x)$.
- $(X_i)_{i=1}^{\infty}$ iidrv with mean μ.
- $S_n = X_1 + \cdots + X_n$.
- $S_n/n \xrightarrow{D} \mu$
- $\phi_{S_n/n}(t) = [\phi_X(t/n)]^n = [1 + it\mu + o(t/n)]^n \to e^{it\mu}$ as $n \to \infty$

Central limit theorem

- Convergence in distribution:
  $X_i \xrightarrow{D} X$ if for all $x \in \mathbb{R}$ such that F is continuous at x,
  $F_{X_n}(x) \to F_X(x)$.
- $(X_i)_{i=1}^{\infty}$ iidrv with mean μ, finite variance $\sigma^2 \neq 0$
- $A_n = \sum (X_i - \mu)/[\sigma\sqrt{n}]$.
- $A_n \xrightarrow{D} N(0,1)$ as $n \to \infty$
- $\phi_{A_n}(t) = [\phi_{(X-\mu)/\sigma}(t/\sqrt{n})]^n$
  $=[1 - \frac{1}{2}t^2/n + o(1/n)]^n \to e^{-\frac{1}{2}t^2} = \phi_{N(0,1)}(t)$ as $n \to \infty$

Cauchy distribution

$$\phi_x(t) = e^{-|t|}$$

$$\phi_{S_n/n}(t) \text{ ???}$$