Lab 2

Questions with a (*) next to them form part of homework 1. Due in 20/10/2014. [Q 1,2,3]

1. (*) St Petersburg Paradox. You are given the option of playing a game. You pay £X to play one round. Each round a fair coin is tossed repeatedly, independently, until a head comes up. If the sequence is

   - H, you win £1,
   - TH, you win £2,
   - TTH, you win £4,
   - TT...TTH, you win £2^n, where n the number of tails.

   (a) Calculate the expected winnings per game.

   (b) For a few different values of X, simulate playing the game repeatedly using R. Plot the trajectory of your net gain (winnings minus stakes). What is the maximum value of X you would be willing to pay in practice?

2. (*) A family has exactly two children:

   (a) If at least one child is a boy, what is the probability the other child is also a boy?

   (b) If at least one child is a boy born on a Monday, what is the probability that the other is also a boy?

   (c) Write a program in R that randomly generates pairs of children. By ignoring some of the pairs, check your answers to part (a) and (b).

   [You can assume children are equally likely to be either gender, and to be born on each day of the week.]

3. (*) You sample $X_1, \ldots, X_{1001}$ items from the Uniform(0,1) distribution. Let $A = \max_{i=1}^{1001} X_i$ denote the maximum value.

   (a) What is the c.d.f of A?

   (b) Check your answer experimentally in R.

   (c) Let $B = \text{median}(X_1, \ldots, X_{1001})$. i.e. sort the X_i and take the 501-th element. What is the distribution of B? Plot the c.d.f. and check your answer experimentally in R.

   (d) Can you approximate the distribution of B using a well known family of distributions?

4. Poisson distribution: The Poisson distribution is defined by $X \sim \text{Poisson}(\lambda)$ if

   $$\mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \ldots$$

   (a) Show that if $(X_i)_{i=1}^n$ are Bernoulli(λ/n) i.i.d.r.v. then $\sum_{i=1}^n X_n \xrightarrow{D} \text{Poisson}(\lambda)$ as $n \to \infty$.

   (b) A long road has cars randomly distributed along it. The total number of cars is distributed according to the Poisson distribution. Each car is either red or blue (the car colors are independent). Show that the number of red cars, and the number of blue cars are both Poisson and independent.

5. The characteristic function of the Binomial distribution $X \sim \text{Bin}(n, p)$ is

   $$\phi(t) = (1 - p + pe^{it})^n = \mathbb{E}(e^{itX}).$$

   Use characteristic functions to show that $Y = (X - np)/\sqrt{n}$ converges in distribution to the normal $N(0, p(1-p))$ distribution. [Recall: $\phi_{aX+b}(t) = e^{itb}\phi_X(at)$].

6. Independence:

(a) Define what it means for two events to be independent.

(b) Define what it means for n events to be independent.

(c) Can an event be independent of itself? [Give an example or show it is impossible.]

(d) Define what it means for two random variables to be independent.

(e) Shop that if two Bernoulli random variables have covariance 0 then they are independent. [Caution! This is not true for general random variables.]

(f) Find events A, B and C such that (i) A and B are independent (ii) B and C are independent (iii) A and C are independent, and (iv) A,B,C are not independent.

(g) Find events A, B and C such that $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$ but A, B and C are not independent. If you can make up a story where A, B and C are natural/interesting events, you get bonus points.