Lab 3

Questions with a (*) next to them form part of homework 1. Due in 20/10/2014. [Q 1]

1. (*) Forensic accounting: Define the "first digit" function f to be the first non-zero digit in a number's decimal representation, e.g. f(123)=1, f(0.0234)=2. Assume that the numbers in a company's accounts can be modeled by the exponential of a normal distribution, say

$$\log(X_i) \sim N(0, \sigma^2), \qquad \sigma = 10$$

with the $(X_i)_{i=1}^n$ i.i.d.r.v.

   (a) What is the distribution of the first digits $f(X_i)$? First try to answer the question experimentally in R. Then try to explain the distribution. [i.e. what is the distribution in the limit $\sigma \to \infty$?]
   (b) Let $Y_k = |\{i : f(X) = k\}|$ for $k = 1, \dots, 9$, i.e. the number of X_i taking value k. What type of distribution is Y_k? What is the standard deviation of Y_k?
   (c) Look up some financial type data, i.e. stock prices. Do they seem to fit the pattern? Does the deviation from the pattern exceed what you would expect based on the distribution of the Y_k.
   (d) Make up a bunch of random numbers. What is the distribution of the first digits for your sample? Does it seem to fit the pattern?

2. Show that:

   (a) If random variables X and Y are independent, then $\text{Cov}(X, Y) = 0$.
   (b) Find random variables X, Y such that $Cov(X, Y) = 0$ but X and Y are not independent.

3. Using characteristic functions, prove the weak law of large numbers and the central limit theorem. [See Casella&Berger, Theorem 2.6.1]

   - The characteristic function of a random variable X is $\phi_X(t) = \mathbb{E}[e^{itX}]$.
   - If X, Y are independent, and $a, b \in \mathbb{R}$, then $\phi_{aX+bY}(t) = \phi_X(at)\phi_Y(bt)$.
   - For a constant μ, $\phi_\mu(t) = e^{it\mu}$.
   - For the N(0,1) distribution, $\phi_{N(0,1)}(t) = \exp(-t^2/2)$.

4. [Poisson arrival process] For $s > 0$, let N_s denote the number of arrivals in the time interval $[0, s]$. Suppose

   (a) $N_0 = 0$
   (b) $s < t \to N_s$ and $N_t - N_s$ are independent
   (c) N_s and $N_{t+s} - N_t$ are identically distributed
   (d) $\mathbb{P}[N_t = 1] = \lambda t + o(t)$ as $t \to 0$ [$f(t) = o(t)$ as $t \to 0$ if $\lim_{t\to 0} f(t)/t = 0$]
   (e) $\mathbb{P}[N_t > 1] = o(t)$ as $t \to 0$.

   Show that $N_s \sim$Poisson$(s\lambda)$.

5. Let X, Y denote independent Uniform$(0, 1)$ random variables. Let $Z = X + Y$.

   (a) What is the distribution of X conditional on $\{Z = z\}$ for $z \in (0, 2)$?
   (b) What is the covariance of X and Z?
   (c) What is the correlation of X and Z?

6. Exchange paradox. There are two sealed envelopes, one with £X in and one with £2X in. You do not know X. You are allowed to pick an envelope and keep the contents. You have picked an envelope but not opened it yet. Should you switch and pick the other envelope? [If you switch, there is a 50% chance of doubling your money, and a 50% chance of halving it: $0.5 \times 2 + 0.5 \times 0.5 > 1$.]