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Networks and Random Processes

Hand-out 1
Linear Algebra

Consider a square matrix $A \in \mathbb{R}^{n\times n}$ with elements a_{ij}. The **determinant** of the matrix is given by

$$\det(A) = \sum_{\pi\in S_n} \mathrm{sgn}(\pi) \prod_{i=1}^n a_{i\pi(i)} ,$$

where the first sum is over all permutations π of the indices $1,\ldots,n$ with associated signature $\mathrm{sgn}(\pi) \in \{-1,1\}$. A has n complex **eigenvalues** $\lambda_i,\ldots,\lambda_n \in \mathbb{C}$ which are the roots of the **characteristic polynomial**

$$\chi_A(\lambda) = \det(A - \lambda\mathbb{I}_n) = \prod_{i=1}^n (\lambda_i - \lambda) ,$$

which is a polynomial of degree n. If λ_i is an eigenvalue, so is the complex conjugate $\bar\lambda_i$, since χ_A has real coefficients. Furthermore

$$\det A = \prod_{i=1}^n \lambda_i \quad \text{and} \quad \mathrm{Tr}(A) = \sum_{i=1}^n a_{ii} = \sum_{i=1}^n \lambda_i ,$$

where the trace Tr is defined as the sum of the diagonal elements of A.

$|v\rangle \in \mathbb{C}^n$ is **right (column) eigenvector** with eigenvalue $\lambda \in \mathbb{C}$ and $\langle u|$ **left (row) eigenvector** if

$$A|v\rangle = \lambda|v\rangle , \quad \langle u|A = \lambda\langle u| .$$

If all eigenvalues are distinct, the matrix has a complete basis of eigenvectors, which can be chosen to be orthogonal and normalized, i.e.

$$\langle u_i|v_j\rangle = \delta_{ij} .$$

Matrices for which this is not the case exhibit higher multiplicity of eigenvalues, e.g. with characteristic polynomial $(1-\lambda)^2$. In the following we assume that all eigenvalues are distinct.

- If $A = A^T$ is symmetric, then all eigenvalues $\lambda_i \in \mathbb{R}$ are real and the eigenvectors have real entries and 'are equal', in the sense that $\langle u_i|^T = |v_i\rangle$ and form an orthonormal basis of $\mathbb{R}^n$.
- If all eigenvalues are distinct, the matrix $|v_i\rangle\langle u_i| \in \mathbb{R}$ projects a vector $\langle x|$ onto the eigenspace of the corresponding eigenvalue λ_i

  $$\langle x|v_i\rangle\langle u_i| = a_i\langle u_i| \quad \text{with coefficient} \quad a_i = \langle x|v_i\rangle .$$

  A itself can be decomposed as a linear combination of such **projectors**, $A = \sum_{i=1}^n \lambda_i|v_i\rangle\langle u_i|$ For projections we have $|v_i\rangle\langle u_i|\,|v_j\rangle\langle u_j| = \delta_{ij}|v_i\rangle\langle u_i|$ so for powers of A we simply get

  $$A^k = \sum_{i=1}^n \lambda_i^k |v_i\rangle\langle u_i| \quad \text{for all } k \geq 1 .$$

- **Gershgorin theorem.** Every eigenvalue of A lies in at least one **Gershgorin disc**

  $$D(a_{ii}, R_i) \subseteq \mathbb{C},\ i = 1,\ldots,n , \quad \text{where} \quad R_i = \sum_{j\neq i} |a_{ij}| .$$