Networks and Random Processes

Hand-out 2

Generating functions, Poisson processes

For a non-negative, integer-valued random variable X we encode the sequence of probabilities $p_n := \mathbb{P}[X = n]$, $n = 0, 1, \ldots$ in the **probability generating function** of X

$$G_X(s) := \mathbb{E}\big[s^X\big] = \sum_{n=0}^{\infty} p_n\, s^n \ , \quad \text{where } s \in [0,1] \text{ is a dummy variable .}$$

Examples.

- $X \sim \mathrm{Be}(q)$ with $q \in [0, 1]$, i.e. $X \in \{0, 1\}$ is a **Bernoulli rv** with $\mathbb{P}[X{=}1]{=}q{=}1{-}\mathbb{P}[X{=}0]$,

$$G_X(s) = (1-q)s^0 + qs = 1 - q + qs \ .$$

- $X \sim \mathrm{Geo}(q)$ with $q \in [0, 1]$, i.e. $X \in \mathbb{N}_0$ is a **geometric rv** with $\mathbb{P}[X = n] = q(1-q)^n$,

$$G_X(s) = \sum_{n=0}^{\infty} q(1-q)^n s^n = \frac{q}{1-(1-q)s} \ .$$

Useful properties.

- Given a generating function $G_X(s)$, we can recover the probabilities p_n by differentiation

$$p_0 = G_X(0) \ , \quad p_1 = G'_X(0) \ , \quad p_2 = \frac{1}{2}\, G''_X(0) \ , \quad \ldots \quad p_n = \frac{1}{n!}\, G_X^{(n)}(0) \ .$$

- Moments of the random variable X are given by derivatives,

$$G_X(1) = 1 \ , \quad G'_X(1) = \mathbb{E}(X) \quad \text{and} \quad \mathrm{Var}(X) = G''_X(1) + G'_X(1) - \big(G'_X(1)\big)^2 \ .$$

- If X, Y are independent non-negative, integer-valued random variables, then

$$G_{X+Y}(s) = G_X(s)\, G_Y(s) \ .$$

  This is usually much easier than evaluating the convolution sum directly

$$\mathbb{P}(X + Y = n) = \sum_{k=0}^{n} \mathbb{P}(X = k)\, \mathbb{P}(Y = n-k) \ .$$

Example. Let $X_1, \ldots, X_n \sim \mathrm{Be}(q)$ be iid Bernoulli. Then $Y = \sum_{i=1}^{n} X_i \sim \mathrm{Bi}(n, q)$ is a **Binomial rv** with $\mathbb{P}[Y = k] = \binom{n}{k} q^k (1-q)^{n-k}$, since with the binomial formula we have

$$G_Y(s) = \big(G_{X_i}(s)\big)^n = \big(1 - q + qs\big)^n = \sum_{k=0}^{n} s^k q^k (1-q)^{n-k} \binom{n}{k} \ .$$

Poisson random variables.
Let $X \sim \mathrm{Poi}(\lambda)$ be a **Poisson** random variable with **intensity** $\lambda \geq 0$, i.e.

$$\mathbb{P}[X = k] = \frac{\lambda^k}{k!}\, e^{-\lambda} \quad \text{for all } k \in \mathbb{N}_0\ .$$

We have $\mathbb{E}[X] = \lambda$, $\mathrm{Var}[X] = \lambda$ and the probability generating function of X is

$$G_X(s) = \mathbb{E}[s^X] = \sum_{k=0}^{\infty} s^k \frac{\lambda^k}{k!}\, e^{-\lambda} = e^{\lambda(s-1)}\ .$$

Therefore, if $X_i \sim \mathrm{Poi}(\lambda_i)$, $i = 1, \ldots, n$ are independent Poisson, then the sum is also Poisson,

$$S = \sum_{i=1}^{n} X_i \sim \mathrm{Poi}(\lambda_1 + \ldots + \lambda_n)\ .$$

For $\alpha \in [0, 1]$, an α**-thinning** $\alpha \circ X$ of an integer random variable $X \in \mathbb{N}_0$ is defined as

$$\alpha \circ X = \sum_{k=1}^{X} Z_k \quad \text{with} \quad Z_k \sim \mathrm{Be}(\alpha) \in \{0, 1\} \quad \text{iid Bernoulli}\ .$$

For Poisson variables we have $\quad X \sim \mathrm{Poi}(\lambda),\ \alpha \in [0, 1] \quad \Rightarrow \quad \alpha \circ X \sim \mathrm{Poi}(\alpha\lambda)\ .$
This follows directly from computing the generating function

$$G_{\alpha \circ X}(s) = \mathbb{E}\big(e^{\sum_{k=1}^{X} Z_k}\big) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\, e^{-\lambda} \mathbb{E}(s^{Z_k})^n = G_X(G_Z(s)) = e^{\lambda\alpha(s-1)}$$

where we have used $G_Z(s) = 1 - \alpha + \alpha s = 1 + \alpha(s - 1)$.

Poisson processes.
A **Poisson process** $N = (N_t : t \geq 0) \sim \mathrm{PP}(\lambda)$ with **rate** $\lambda > 0$ is a Markov chain with independent stationary increments, and $N_t \sim \mathrm{Poi}(\lambda t)$ for all $t \geq 0$. We know from lectures that the holding times of the chain are independent $\mathrm{Exp}(\lambda)$ variables with mean $1/\lambda$. The above properties for Poisson random variables imply the following for processes:

- **Adding Poisson processes.**
  Let $N^i \sim \mathrm{PP}(\lambda_i)$ be independent Poisson processes, and define their sum $M = (M_t : t \geq 0)$ via $M_t := N_t^1 + \ldots + N_t^n$ for all $t \geq 0$. Then $M \sim \mathrm{PP}(\lambda_1 + \ldots + \lambda_n)$ is a Poisson process.

- Let $\tau_1, \ldots, \tau_n$ be independent $\mathrm{Exp}(\lambda_i)$ **exponential** random variables, corresponding to the holding times of Poisson processes N^i. Then

  $$\min\{\tau_1, \ldots, \tau_n\} \sim Exp(\lambda_1 + \ldots + \lambda_n)\ ,$$

  corresponding to the holding time of the process $N^1 + \ldots + N^n$.

- **Thinning.**
  An α-thinning $\alpha \circ N$ of a Poisson process $N \sim \mathrm{PP}(\lambda)$ is defined via $(\alpha \circ N)_t = \alpha \circ N_t$ for all $t \geq 0$, i.e. independently keep jumps with probability α and erase the others.
  Then $\alpha \circ N \sim \mathrm{PP}(\alpha\lambda)$ is again a Poisson process.