Networks and Random Processes

Problem sheet 1

Sheet counts $40/100$ homework marks, [x] indicates weight of the question.
Please put solutions in my pigeon hole or give them to me by **Wednesday, 14.10.2015, 4pm**.

1.1 Random walk I [6]

(a) Consider a sequence of independent coin tosses with $\mathbb{P}[\text{heads}] = p$ and let Y_n be the number of heads up to time n with $Y_0 = 0$.
Give the state space S and transition matrix for the process $(Y_n : n \in \mathbb{N}_0)$.
Compute $\mathbb{P}[Y_n = k]$ for all $n, k \geq 0$ using the binomial formula.

(b) Let $(Z_n : n \in \mathbb{N}_0)$ be a simple random walk on $S = \mathbb{Z}$ with transition matrix $p(x, y) = p\delta_{y,x+1} + q\delta_{y,x-1}$.
Use that $Z_n = 2Y_n - n$ to get a formula for $\mathbb{P}[Z_n = k]$ for all $n, k \geq 0$.
Does $(Z_n : n \in \mathbb{N}_0)$ have a stationary distribution on $\mathbb{Z}$?

(c) Stirling's formula says that

$$n! = \Big(\frac{n}{e}\Big)^n \sqrt{2\pi n}\big(1 + o(1)\big) \quad \text{as } n \to \infty \ .$$

Use this to derive the asymptotic behaviour of $\mathbb{P}[Z_{2n} = 0]$ as $n \to \infty$.

1.2 Random walk II [10]

(a) Consider a simple random walk on $\{1, \ldots, L\}$ with probabilities $p \in [0, 1]$ and $q = 1 - p$ to jump right and left, respectively, and with periodic and closed boundary conditions. In each case, sketch the transition matrix P of the process (see lectures), decide whether the process is irreducible, and give all stationary distributions $\boldsymbol{\pi}$ and state whether they are reversible. (Hint: Use detailed balance.)
Discuss the case $p = 1$ and $p = q = 1/2$.

(b) Consider the same SRW with absorbing boundary conditions, sketch the transition matrix P, decide whether the process is irreducible, and give all stationary distributions $\boldsymbol{\pi}$ and state whether they are reversible.
Let $h_k^L = \mathbb{P}[X_n = L \text{ for some } n \geq 0 | X_0 = k]$ be the absorption probability in site L.
Give a recursion formula for h_k^L and solve it.

(c) Consider a tree, i.e. an undirected, connected graph (G, E) without loops and double edges. A simple random walk on (G, E) has transition probabilities

$$p(x, y) = e(x, y)/d(x) \text{ for } d(x) > 0 \quad \text{and} \quad p(x, y) = \delta_{x,y} \text{ for } d(x) = 0 \ ,$$

where $e(x, y) = e(y, x) \in \{0, 1\}$ denotes the presence of an undirected edge (x, y), and $d(x) = \sum_{y \in G} e(x, y)$ is the degree of vertex x.
Find a formula for the stationary distribution $\boldsymbol{\pi}$.

1.3 Generators and eigenvalues [8]

Consider the continuous-time Markov chain $(X_t : t \geq 0)$ with generator $G = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -4 & 3 \\ 0 & 1 & -1 \end{pmatrix}$.

(a) Draw a graph representation for the chain (i.e. connect the three states by their jump rates), and give the transition matrix P^Y of the corresponding jump chain $(Y_n : n \in \mathbb{N}_0)$.

(b) Consider the Taylor series of the matrix P_t and convince yourself that $\frac{d}{dt} P_t|_{t=0} = G$, $\frac{d}{dt} P_t|_{t=0} = G^2$ etc..
Assume that $G = B^{-1} \Lambda B$, with diagonal matrix $\Lambda \in \mathbb{R}^{3\times 3}$ and eigenvalues λ_i of G on the diagonal, and with some matrix $B \in \mathbb{R}^{3\times 3}$. Show that

$$P(t) = \exp(tG) = B^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} B\,.$$

It is not necessary to compute entries of the matrix B.

(c) Compute λ_2 and λ_3. Use this to compute $p_{11}(t)$, i.e. determine the coefficients in

$$p_{11}(t) = a + b\, e^{\lambda_2 t} + c\, e^{\lambda_3 t}\,.$$

(Hint: Use the first part of (b), it is again not necessary to compute the matrix B.)

(d) What is the stationary distribution $\boldsymbol{\pi}$ of X?

1.4 Toom's model (A probabilistic cellular automaton) [16]

Consider a fixed population of L individuals on a one-dimensional lattice $\Lambda = \{1, \ldots, L\}$ with periodic boundary conditions. At time $t = 0$ each site i is occupied by a type $X_0(i) \in \{1, \ldots, N\}$, where N is the total number of types. Time is counted in discrete generations $t = 0, 1, \ldots$, and the lattice at odd times is shifted by $1/2$. So in generation $t+1$ each individual has two parents, from one of which it inherits the type. The type $X_{t+1}(i + 1/2)$ is determined by $X_t(i)$ and $X_t(i+1)$ according to the following rules, for $x \neq y \in \{1, \ldots, N\}$:

$$xx \to x\,, \quad yy \to y\,, \quad xy, yx \to \begin{cases} x\,, & \text{with prob. } \hat{p}_{xy} \\ y\,, & \text{with prob. } \hat{p}_{yx} = 1 - \hat{p}_{xy} \end{cases}.$$

We focus on $\hat{p}_{xy} = 1/2$ for all x, y, i.e. all types have the same 'fitness'.

(a) Simulate the dynamics of this process (e.g. using MATLAB) up to generation T in a $T \times 2L$ matrix (with time growing upwards). Initialize the matrix with 0s, then fill the even sites in the bottom row with initial condition $X_0(i) = i$ (i.e. a different type on each site). Then assign the odd sites in the next row until all rows are filled.
Visualise the matrix using e.g. the MATLAB function 'image'.
(Make sure that the empty sites show up in white or a very light colour, you might have to replace 0 by another (negative) value.)
You may use the suggested parameter values $L = 100$, $T = 500$ or any other that make sense (it is a good idea to vary them to get a feeling for the model). Address the following points, supported by appropriate visualisations:
- Explain the emerging patterns in a couple of sentences.
- What are the stationary distributions for the process X_t, i.e. what will happen when you run the simulation long enough?
- How long will it roughly take to reach stationarity (depending on L)? Test your answer using three values for L, e.g. 10, 50 and 100.

(b) Let N_t be the number of individuals of a given species at generation t, using the same initial condition as in (a), i.e. $N_0 = 1$. Show that $(N_t : t \in \mathbb{N})$ is actually a random walk, give the state space and the transition probabilities.
Is the walk irreducible? What is the stationary distribution provided that $N_0 = 1$?
Using a recursion as in Q1.2(b), compute the expected hitting time of the absorbing states.