Networks and Random Processes

Problem sheet 2

Sheet counts $60/100$ homework marks, [x] indicates weight of the question.
Please put solutions in my pigeon hole or give them to me by **Wednesday, 04.11.2015, 23:59**.

2.1 Birth-death processes [6]

A birth-death process $(X_t : t \geq 0)$ is a continuous-time Markov chain with state space $S = \mathbb{N}_0 = \{0, 1, \ldots\}$ and jump rates

$$x \xrightarrow{\alpha_x} x+1 \quad \text{for all } x \in S\ , \qquad x \xrightarrow{\beta_x} x-1 \quad \text{for all } x \geq 1\ .$$

(a) Write the generator G and the master equation. Under which conditions on the rates is X irreducible?
Using detailed balance, find a formula for the stationary probablities $\pi(x)$ in terms of $\pi(0)$.

(b) Suppose $\alpha_x = \alpha$ for $x \geq 0$ and $\beta_x = \beta$ for $x > 0$. This is called an $M/M/1$ **queue**.
Under which conditions on α and β can the stationary distribution be normalized? Give a formula for $\pi(x)$ in that case.

2.2 Contact process [15]

Consider the CP $(\eta_t : t \geq 0)$ on the complete graph $\Lambda = \{1, \ldots, L\}$ (i.e. $q(i,j) = \lambda$ for all $i \neq j$) with state space $S = \{0,1\}^L$ and transition rates

$$c(\eta, \eta^i) = \eta(i) + \lambda\big(1 - \eta(i)\big) \sum_{j \neq i} \eta(j)\ ,$$

where $\eta, \eta^i \in S$ are connected states such that $\eta^i(k) = \begin{cases} 1 - \eta(k) & , \ k = i \\ \eta(k) & , \ k \neq i \end{cases}$,
(η with state of individual i flipped).

(a) Let $N_t = \sum_{i \in \Lambda} \eta_t(i) \in \{0, \ldots, L\}$ be the number of infected individuals at time t.
Show that $(N_t : t \geq 0)$ is a Markov chain with state space $\{0, \ldots, L\}$ by computing the transition rates $g(n,m)$ for $n, m \in \{0, \ldots, L\}$.
Write down the master equation for the process $(N_t : t \geq 0)$.

(b) Is the process $(N_t : t \geq 0)$ irreducible, does it have absorbing states?
What are the stationary distributions?

(c) Assume that $\mathbb{E}\big(N_t^k\big) = \mathbb{E}(N_t)^k$ for all $k \geq 1$. This is called a **mean-field assumption**, meaning basically that we replace the random variable N_t by its expected value.
Use this assumption to derive the **mean-field rate equation** for $\rho(t) := \mathbb{E}(N_t)/L$,

$$\frac{d}{dt}\,\rho(t) = f\big(\rho(t)\big) = -\rho(t) + L\lambda\big(1 - \rho(t)\big)\rho(t)\ .$$

(d) Analyze this equation by finding the stable and unstable stationary points via $f(\rho^*) = 0$.
What is the prediction for the stationary density ρ^* depending on λ?

2.3 Simulation of CP (Sample code on the course webpage) [15]

Consider the contact process $(\eta_t : t \geq 0)$ as defined in Q2.2, but now on the one-dimensional lattice $\Lambda_L = \{1, \ldots, L\}$ with connections only between nearest neighbours, i.e. $q(i,j) = q(j,i) = \lambda\delta_{j,i+1}$, and periodic boundary conditions.
The critical infection rate λ_c can be defined such that the infection on the infinite lattice $\Lambda = \mathbb{Z}$ started from the fully infected lattice dies out for $\lambda < \lambda_c$, and survives for $\lambda > \lambda_c$. It is known numerically up to several digits, depends on the dimension, and lies in $[1, 2]$ in our case.
All simulations of the process should be done with initial condition $\eta_0(i) = 1$ for all $i \in \Lambda$.

(a) To get a general idea, simulate the process for e.g. $L = 256$ for several values of $\lambda \in [1, 2]$. Plot the number of infected individuals $N_t = \sum_{x\in\Lambda_L} \eta_t(x)$ as a function of time up to time $10 \times L$, averaging over 100 realizations in a double-logarithmic plot to find the window of interest for the parameter λ, choosing e.g. $\lambda = 1, 1.2, \ldots, 1.8, 2$.
What is the expected behaviour of N_t depending on λ?

(b) Then use fine increments of 0.01 for λ and averages of at least 500 realizations to find an estimate of the critical value $\lambda_c(L) \in [1, 2]$.
Repeat this for different lattice sizes, e.g. $L = 128, 256, 512, 1024$, and plot your estimates of $\lambda_c(L)$ against $1/L$. Extrapolate to $1/L \to 0$ to get an estimate of $\lambda_c = \lambda_c(\infty)$.
This approach is called **finite size scaling**, in order to correct for systematic **finite size effects** which influence the critical value.

(c) Explain how to set up an update loop that can be used to simulate the contact process on a general connected, undirected graph.

2.4 Dorogovtsev-Mendes-Samukhin model [12]

Consider the following generalization of the Barabási-Albert model. Starting with $m_0 = 5$ connected nodes, in each timestep a node j is added and linked to $m = 5$ existing distinct nodes according to the probability (to be adapted to avoid double edges)

$$\pi_{j\leftrightarrow i} = \frac{k_0 + k_i}{\sum_{i\in V(t)}(k_i + k_0)}\ , \quad k_0 \in \mathbb{N}_0\ .$$

Simulate the model for three different values of $k_0 = 0, 2, 4$ to generate graphs of size $N = |V| = 1000$, with 20 independent realizations in each case.

(a) Plot the tail of the degree distribution in a double logarithmic plot for a single realization and for all 100. For each k_0 compare the tail to the power law with exponent $-2 - k_0/m$.

(b) Compute $k_{nn}(k) = \mathbb{E}\Big[\sum_{i\in V} k_{nn,i}\delta_{k_i,k} \Big/ \sum_{i\in V}\delta_{k_i,k}\Big]$ where $k_{nn,i} = \frac{1}{k_i}\sum_{j\in V} a_{ij}k_j$, and decide whether the graphs are typically uncorrelated or assortative.

(c) Plot the spectrum of the adjacency matrix A using all realizations with `ksdensity`, and compare it to the Wigner semi-circle law.

2.5 Erdős Rényi random graphs [12]

Consider the Erdős Rényi random graph model and simulate at least 20 realizations of $\mathcal{G}_{N,p}$ graphs with $p = p_N = z/N$, $z = 0.1, 0.2, \ldots, 3.0$ for $N = 100$ and $N = 1000$.

(a) Plot the expected size of the largest two components against z for both values of N.

(b) For $N = 1000$ plot the expected local clustering coefficient $\mathbb{E}[\langle C_i\rangle]$ against z.

(c) Consider now $z = 0.5, 1.5, 5$ and 10. Plot the spectrum of the adjacency matrix A using all realizations with `ksdensity`, and compare it to the Wigner semi-circle law.