# MA933 - Networks and Random Processes

## MSc in Mathematics of Systems

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These notes and other information about the course are available on

`www2.warwick.ac.uk/fac/sci/mathsys/courses/msc/ma933/`

# Contents



## References


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- C.W. Gardiner: Handbook of Stochastic Methods (3rd edition), Springer 2004
- G. Grimmett: Probability on Graphs, CUP 2010
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# 1. Probability

- **probability space** $\Omega$ $\quad\big($ e.g. $\{H, T\}$, $\{$paths of a stoch. process$\}\big)$
- **events** $A \subseteq \Omega$ $\quad$ (measurable) subsets $\quad$ (e.g. odd numbers on a die)

  $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is the set of all events (subset of the powerset)

## Definition 1.1

A **probability distribution** $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ which is

(i) normalized, i.e. $\quad \mathbb{P}[\emptyset] = 0 \quad$ and $\quad \mathbb{P}[\Omega] = 1$

(ii) additive, i.e. $\quad \mathbb{P}\big[\cup_i A_i\big] = \sum_i \mathbb{P}[A_i]$ ,

where $A_1, A_2, \ldots$ is a collection of disjoint events, i.e. $A_i \cap A_j = \emptyset$ for all $i, j$.

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **probability space**.

- For **discrete** $\Omega$: $\quad \mathcal{F} = \mathcal{P}(\Omega) \quad$ and $\quad \mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega]$

  e.g. $\quad \mathbb{P}[\text{even number on a die}] = \mathbb{P}[2] + \mathbb{P}[4] + \mathbb{P}[6] = 1/2$
- For **continuous** $\Omega$ $\quad$ (e.g. $[0, 1]$): $\quad \mathcal{F} \subsetneq \mathcal{P}(\Omega)$

# 1. Independence and conditional probability

- Two events $A, B \subseteq \Omega$ are called **independent** if $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$ .

  **Example.** rolling a die repeatedly
- If $\mathbb{P}[B] > 0$ then the **conditional probability** of $A$ given $B$ is

$$\mathbb{P}[A|B] := \mathbb{P}[A \cap B]/\mathbb{P}[B] .$$

  If $A$ and $B$ are independent, then $\mathbb{P}[A|B] = \mathbb{P}[A]$.

## Lemma 1.1 (Law of total probability)

Let $B_1, \ldots, B_n$ be a **partition** of $\Omega$ such that $\mathbb{P}[B_i] > 0$ for all $i$. Then

$$\mathbb{P}[A] = \sum_{i=1}^{n} \mathbb{P}[A \cap B_i] = \sum_{i=1}^{n} \mathbb{P}[A|B_i]\,\mathbb{P}[B_i] .$$

Note that also $\mathbb{P}[A|C] = \sum_{i=1}^{n} \mathbb{P}[A|C \cap B_i]\,\mathbb{P}[B_i|C]$ .

# 1. Random variables

## Definition 1.2

A **random variable** $X$ is a (measurable) function $X : \Omega \to \mathbb{R}$.

The **distribution function** of the random variable is

$$F(x) = \mathbb{P}[X \leq x] = \mathbb{P}\big[\{\omega : X(\omega) \leq x\}\big] \ .$$

$X$ is called **discrete**, if it only takes values in a countable subset $\{x_1, x_2, \ldots\}$ of $\mathbb{R}$, and its distribution is characterized by the **probability mass function**

$$\pi(x_k) := \mathbb{P}[X = x_k] \ , \quad k = 1, 2, \ldots \ .$$

$X$ is called **continuous**, if its distribution function is

$$F(x) = \int_{-\infty}^{x} f(y)\, dy \quad \text{for all } x \in \mathbb{R} \ ,$$

where $f : \mathbb{R} \to [0, \infty)$ is the **probability density function (PDF)** of $X$.

# 1. Random variables

- In general, $f = F'$ is given by the derivative (exists for cont. rv's).
  For discrete rv's, $F$ is a step function with 'PDF'

$$f(x) = F'(x) = \sum_k \pi(x_k)\delta(x - x_k) \ .$$

- The **expected value** of $X$ is given by

$$\mathbb{E}[X] = \begin{cases} \sum_\omega X(\omega)\,\mathbb{P}[\omega] = \sum_k x_k \pi(x_k) \\ \int_\Omega X(\omega) d\mathbb{P}(\omega) = \int_{\mathbb{R}} x f(x)\,dx \end{cases}$$

- The **variance** is given by $\quad \mathrm{var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$
- Two random variables $X, Y$ are independent if the events $\{X \leq x\}$ and $\{Y \leq y\}$ are independent for all $x, y \in \mathbb{R}$. This implies for **joint distributions**

$$f(x,y) = f_X(x) f_Y(y) \quad \text{or} \quad \pi(x_k, x_l) = \pi^X(x_k)\,\pi^Y(x_l)$$

  where $\quad f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy \quad$ and $\quad \pi^X(x_k) = \sum_l \pi(x_k, x_l) \quad$ are the **marginals**.

**Example.** (successive) coin tosses with $\Omega = \{H, T\}$ and $X(H) = -1, X(T) = 1$

# 1. Simple random walk

## Definition 1.3

Let $X_1, X_2, \ldots \in \{-1, 1\}$ be a sequence of independent, identically distributed random variables **(iidrv's)** with

$$p = \mathbb{P}[X_i = 1] \quad \text{and} \quad q = \mathbb{P}[X_i = -1] = 1 - p \, .$$

The sequence $Y_0, Y_1, \ldots$ defined as $\quad Y_0 = 0 \quad$ and $\quad Y_n = \sum_{k=1}^{n} X_k$

is called the **simple random walk (SRW)** on $\mathbb{Z}$.

- for a single **increment** $X_k$ we have

$$\mathbb{E}[X_k] = p - q = 2p - 1 \, , \quad \operatorname{var}[X_k] = p + q - (p - q)^2 = 4p(1 - p)$$

- $\mathbb{E}[Y_n] = \mathbb{E}\left[\sum_{k=1}^{n} X_k\right] = \sum_{k=1}^{n} \mathbb{E}[X_k] = n(2p - 1)$
  (expectation is a linear operation)
- $\operatorname{var}[Y_n] = \operatorname{var}\left[\sum_{k=1}^{n} X_k\right] = \sum_{k=1}^{n} \operatorname{var}[X_k] = 4np(1 - p)$
  (for a sum of **independent** rv's the variance is additive)

# 1. LLN and CLT

## Theorem 1.2 (Weak law of large numbers (LLN))

Let $X_1, X_2, \ldots \in \mathbb{R}$ be a sequence of iidrv's with $\mu := \mathbb{E}[X_k] < \infty$ and $\mathbb{E}[|X_k|] < \infty$. Then

$$\frac{1}{n} Y_n = \frac{1}{n} \sum_{k=1}^{n} X_k \to \mu \quad \text{as } n \to \infty$$

in distribution (i.e. the distr. fct. of $Y_n$ converges to $\mathbb{1}_{[\mu,\infty)}(x)$ for $x \neq \mu$) .

## Theorem 1.3 (Central limit theorem (CLT))

Let $X_1, X_2, \ldots \in \mathbb{R}$ be a sequence of iidrv's with $\mu := \mathbb{E}[X_k] < \infty$ and $\sigma^2 := \mathrm{var}[X_k] < \infty$. Then

$$\frac{Y_n - n\mu}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (X_k - \mu) \to \xi \quad \text{as } n \to \infty$$

in distr., where $\xi \sim N(0, \sigma^2)$ is a **Gaussian** with PDF $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)}$ .

**Expansion.** as $n \to \infty$, $Y_n = n\mu + \sqrt{n}\sigma\xi + o(\sqrt{n})$ , $\xi \sim N(0,1)$

# 1. Discrete-time Markov processes

## Definition 1.4

A **discrete-time stochastic process** with **state space** $S$ is a sequence $Y_0, Y_1, \ldots = (Y_n : n \in \mathbb{N}_0)$ of random variables taking values in $S$.

The process is called **Markov**, if for all $A \subseteq S$, $n \in \mathbb{N}_0$ and $s_0, \ldots, s_n \in S$

$$\mathbb{P}(Y_{n+1} \in A | Y_n = s_n, \ldots, Y_0 = s_0) = \mathbb{P}(Y_{n+1} \in A | Y_n = s_n) \ .$$

A Markov process (MP) is called **homogeneous** if for all $A \subseteq S$, $n \in \mathbb{N}_0$ and $s \in S$

$$\mathbb{P}(Y_{n+1} \in A | Y_n = s) = \mathbb{P}(Y_1 \in A | Y_0 = s) \ .$$

If $S$ is discrete, the MP is called a **Markov chain (MC)**.

The generic probability space $\Omega$ is the **path space**

$$\Omega = D(\mathbb{N}_0, S) := S^{\mathbb{N}_0} = S \times S \times \ldots$$

which is uncountable even when $S$ is finite. For a given $\omega \in \Omega$ the function $n \mapsto Y_n(\omega)$ is called a **sample path**.

Up to finite time $N$ and with finite $S$, $\Omega_N = S^N$ is finite.

# 1. Discrete-time Markov processes

**Examples.**

- For the simple random walk we have state space $S = \mathbb{Z}$ and $Y_0 = 0$. Up to time $N$, $\mathbb{P}$ is a distribution on the finite path space $\Omega_N$ with

$$\begin{aligned}\mathbb{P}(\omega) &= \prod_{n=1}^{N} \left(p\,\delta_{1,Y_n(\omega)-Y_{n-1}(\omega)} + q\,\delta_{-1,Y_n(\omega)-Y_{n-1}(\omega)}\right) \\ &= p^{\#\text{ of up-steps}}\, q^{\#\text{ of down-steps}} \qquad (= (1/2)^N \text{ for } p = q = 1/2)\,.\end{aligned}$$

  There are only $2^N$ paths in $\Omega_N$ with non-zero probability.
- For the generalized random walk with $Y_0 = 0$ and increments $Y_{n+1} - Y_n \in \mathbb{R}$, we have $S = \mathbb{R}$ and $\Omega_N = \mathbb{R}^N$ with an uncountable number of possible paths.
- A sequence $Y_0, Y_1, \ldots \in S$ of iidrv's is also a Markov process with state space $S$.
- Let $S = \{1, \ldots, 52\}$ be a deck of cards, and $Y_1, \ldots, Y_{52}$ be the cards drawn at random without replacement. Is this a Markov process?

## Proposition 1.4

Let $(X_n : n \in \mathbb{N}_0)$ by a homogeneous DTMC with state space $S$. Then the **transition function**

$$p_n(x,y) := \mathbb{P}[X_n = y | X_0 = x] = \mathbb{P}[X_{k+n} = y | X_k = x] \quad \text{for all } k \geq 0$$

is well defined and fulfills the **Chapman Kolmogorov equations**

$$p_{n+k}(x,y) = \sum_{z \in S} p_n(x,z)\, p_k(z,y) \quad \text{for all } k,n \geq 0,\ x,y \in S\,.$$

**Proof.** We use the law of total probability, the Markov property and homogeneity

$$\begin{aligned}
\mathbb{P}[X_{n+k} = y | X_0 = x] &= \sum_{z \in S} \mathbb{P}[X_{n+k} = y | X_k = z,\, X_0 = x]\, \mathbb{P}[X_k = z | X_0 = x] \\
&= \sum_{z \in S} \mathbb{P}[X_{n+k} = y | X_k = z]\, \mathbb{P}[X_k = z | X_0 = x] \\
&= \sum_{z \in S} \mathbb{P}[X_n = y | X_0 = z]\, \mathbb{P}[X_k = z | X_0 = x]
\end{aligned}$$

# 1. Markov chains

- In matrix form with $P_n = \big(p_n(x,y) : x,y \in S\big)$ the Chapman Kolmogorov equations read

$$P_{n+k} = P_n\, P_k \quad \text{and in particular} \quad P_{n+1} = P_n\, P_1 \ .$$

  With $P_0 = \mathbb{I}$, the obvious solution to this recursion is

$$P_n = P^n \quad \text{where we write} \quad P_1 = P = \big(p(x,y) : x,y \in S\big) \ .$$

- The **transition matrix** $P$ and the initial condition $X_0 \in S$ completely determine a homogeneous DTMC, since for all $k \geq 1$ and all events $A_1, \ldots, A_k \subseteq S$

$$\mathbb{P}[X_1 \in A_1, \ldots, X_k \in A_k] = \sum_{s_1 \in A_1} \cdots \sum_{s_k \in A_k} p(X_0, s_1) p(s_1, s_2) \cdots p(s_{k-1}, s_k) \ .$$

- Fixed $X_0$ can be replaced by an **initial distribution** $\pi_0(x) := \mathbb{P}[X_0 = x]$ .

  The distribution at time $n$ is then

$$\pi_n(x) = \sum_{y \in S} \sum_{s_1 \in S} \cdots \sum_{s_{n-1} \in S} \pi_0(y) p(y, s_1) \cdots p(s_{n-1}, x) \quad \text{or} \quad \boldsymbol{\pi_n = \pi_0 P^n} \ .$$

# 1. Transition matrices

The transition matrix $P$ is **stochastic**, i.e.

$$p(x,y) \in [0,1] \quad \text{and} \quad \sum_y p(x,y) = 1 \ .$$

## Example 1 (Random walk with boundaries)

Let $(X_n : n \in \mathbb{N}_0)$ be a SRW on $S = \{1, \ldots, L\}$ with $p(x,y) = p\delta_{y,x+1} + q\delta_{y,x-1}$.

The boundary conditions are

- **periodic** if $p(L,1) = p \ , \quad p(1,L) = q \ ,$
- **absorbing** if $p(L,L) = 1 \ , \quad p(1,1) = 1 \ ,$
- **closed** if $p(1,1) = q \ , \quad p(L,L) = p \ ,$
- **reflecting** if $p(1,2) = 1 \ , \quad p(L,L-1) = 1 \ .$

# 1. Stationary distributions

## Definition 1.5

Let $(X_n : n \in \mathbb{N}_0)$ be a homogeneous DTMC with state space $S$. The distribution $\pi(x),\ x \in S$ is called **stationary** if for all $y \in S$

$$\sum_{x \in S} \pi(x) p(x,y) = \pi(y) \quad \text{or} \quad \boldsymbol{\pi} P = \boldsymbol{\pi} \ .$$

$\pi$ is called **reversible** if it fulfills the **detailed balance** conditions

$$\pi(x) p(x,y) = \pi(y) p(y,x) \quad \text{for all } x, y \in S \ .$$

- reversibility implies stationarity, since

$$\sum_{x \in S} \pi(x) p(x,y) = \sum_{x \in S} \pi(y) p(y,x) = \pi(y) \ .$$

- Stationary distributions are left **eigenvectors** with **eigenvalue** 1 .

# 1. Distribution at time $n$

Consider a DTMC on a finite state space with $|S| = L$, and let $\lambda_1, \ldots, \lambda_L \in \mathbb{C}$ be the **eigenvalues** of the transition matrix $P$ with corresponding

$$\textbf{left (row) eigenvectors } \langle u_i| \quad \text{and} \quad \textbf{right (column) eigenvectors } |v_i\rangle$$

in bra-ket notation. Assuming that **all eigenvalues are distinct** we have

$$A = \sum_{i=1}^{L} \lambda_i |v_i\rangle\langle u_i| \quad \text{and} \quad A^n = \sum_{i=1}^{L} {\lambda_i}^n |v_i\rangle\langle u_i|$$

since eigenvectors can be chosen **orthonormal** $\langle u_i|v_j\rangle = \delta_{i,j}$.

Since $\boldsymbol{\pi}_n = \boldsymbol{\pi}_0 P^n$ we get

$$\langle \boldsymbol{\pi}_n| = \langle \boldsymbol{\pi}_0|v_1\rangle {\lambda_1}^n \langle u_1| + \ldots + \langle \boldsymbol{\pi}_0|v_L\rangle {\lambda_L}^n \langle u_L| \, .$$

- The **Gershgorin theorem** implies that $|\lambda_i| \leq 1$ and the dependence on the initial condition $\boldsymbol{\pi}_0$ decays exponentially in directions where $|\lambda_i| < 1$.
- $\lambda_1 = 1$ corresponds to the stationary distribution and $|v_1\rangle = (1, \ldots, 1)^T$.
- Other eigenvalues with $|\lambda_i| = 1$ and $\lambda_i \neq 1$ correspond to persistent oscillations.

# 1. Lazy Markov chains

## Definition 1.6

Let $(X_n : n \in \mathbb{N}_0)$ be a DTMC with transition matrix $p(x, y)$. The DTMC with transition matrix

$$p^\epsilon(x, y) = \epsilon\delta_{x,y} + (1 - \epsilon)\, p(x, y)\ , \quad \epsilon \in (0, 1)$$

is called a **lazy version** of the original chain.

- Since all diagonal elements are bounded below by $\epsilon > 0$, the Gershgorin theorem now implies for the eigenvalues of $P^\epsilon$

$$|\lambda_i| = 1 \quad \Rightarrow \quad \lambda_i = 1\ .$$

  Such a matrix $P^\epsilon$ is called **aperiodic**, and there are no persistent oscillations.
- The stationary distribution is unique if and only if the eigenvalue $\lambda = 1$ has multiplicity 1, which is independent of lazyness and is discussed later.

# 1. Absorbing states

## Definition 1.7

A state $s \in S$ is called **absorbing** for a DTMC with transition matrix $p(x,y)$, if

$$p(s,y) = \delta_{s,y} \quad \text{for all } y \in S\,.$$

**RW with absorbing BC.**

Let $h_k$ be the **absorption probability** for $X_0 = k \in S = \{1,\dots,L\}$,

$$h_k = \mathbb{P}[\text{absorption}|X_0 = k] = \mathbb{P}[X_n \in \{1,L\} \text{ for some } n \geq 0 | X_0 = k]\,.$$

Conditioning on the first jump and using Markov, we have the recursion

$$h_k = p h_{k+1} + q h_{k-1} \quad \text{for } k = 2,\dots,L-1\,; \quad h_1 = h_L = 1\,.$$

**Ansatz for solution** $h_k = \lambda^k\,, \quad \lambda \in \mathbb{C}$:

$$\lambda = p\lambda^2 + q \quad \Rightarrow \quad \lambda_1 = 1\,, \quad \lambda_2 = q/p$$

**General solution** of 2nd order linear recursion

$$h_k = a\lambda_1^k + b\lambda_2^k = a + b(q/p)^k\,, \quad a,b \in \mathbb{R}\,.$$

Determine coefficients from boundary condition $\Rightarrow$ $h_k \equiv 1$

# 2. Continuous-time Markov chains

## Definition 2.1

A **continuous-time stochastic process** with **state space** $S$ is a family $(X_t : t \geq 0)$ of random variables taking values in $S$. The process is called **Markov**, if for all $A \subseteq S$, $n \in \mathbb{N}$, $t_1 < \ldots < t_{n+1} \in [0, \infty)$ and $s_1, \ldots, s_n \in S$

$$\mathbb{P}(X_{t_{n+1}} \in A | X_{t_n} = s_n, \ldots, X_{t_1} = s_1) = \mathbb{P}(X_{t_{n+1}} \in A | X_{t_n} = s_n) \ .$$

A Markov process (MP) is called **homogeneous** if for all $A \subseteq S$, $t, u > 0$ and $s \in S$

$$\mathbb{P}(X_{t+u} \in A | X_u = s) = \mathbb{P}(X_t \in A | X_0 = s) \ .$$

If $S$ is discrete, the MP is called a continuous-time **Markov chain (CTMC)**.

The generic probability space $\Omega$ of a CTMC is the space of **right-continuous paths**

$$\Omega = D([0, \infty), S) := \left\{ X : [0, \infty) \to S \,\middle|\, X_t = \lim_{u \searrow t} X_u \right\}$$

For a given $\omega \in \Omega$ the function $t \mapsto X_t(\omega)$ is called a **sample path**.

# 2. Continuous-time Markov chains

## Proposition 2.1

Let $(X_t : t \geq 0)$ by a homogeneous CTMC with state space $S$. Then for all $t \geq 0$ the **transition function**

$$p_t(x,y) := \mathbb{P}[X_t = y | X_0 = x] = \mathbb{P}[X_{t+u} = y | X_u = x] \quad \text{for all } u \geq 0$$

is well defined and fulfills the **Chapman Kolmogorov equations**

$$p_{t+u}(x,y) = \sum_{z \in S} p_t(x,z)\, p_u(z,y) \quad \text{for all } t,u \geq 0,\ x,y \in S\,.$$

In matrix notation $P_t = \big(p_t(x,y) : x,y \in S\big)$ we get

$$P_{t+u} = P_t\, P_u \quad \text{with} \quad P_0 = \mathbb{I}\,.$$

In particular $\frac{P_{t+\Delta t} - P_t}{\Delta t} = P_t \frac{P_{\Delta t} - \mathbb{I}}{\Delta t} = \frac{P_{\Delta t} - \mathbb{I}}{\Delta t} P_t\,,$

taking $\Delta t \searrow 0$ we get the so-called **forward and backward equations**

$$\frac{d}{dt} P_t = P_t G = G P_t\,, \quad \text{where} \quad G = \left.\frac{dP_t}{dt}\right|_{t=0}$$

is called the **generator** of the process.

# 2. Continuous-time Markov chains

- The solution is given by the matrix exponential

$$P_t = \exp(tG) = \sum_{k=0}^{\infty} \frac{t^k}{k!} G^k = \mathbb{I} + tG + \frac{t^2}{2} G^2 + \dots \tag{2.1}$$

- The distribution $\boldsymbol{\pi}_t$ at time time $t > 0$ is then given by

$$\boldsymbol{\pi}_t = \boldsymbol{\pi}_0 \exp(tG) \quad \text{which solves} \quad \frac{d}{dt}\boldsymbol{\pi}_t = \boldsymbol{\pi}_t G \,. \tag{2.2}$$

- On a finite state space with $\lambda_1, \dots, \lambda_L \in \mathbb{C}$ being eigenvalues of $G$, $P_t$ has eigenvalues $\exp(t\lambda_i)$ with the same eigenvectors $\langle v_i|$, $|u_i\rangle$.

  If the $\lambda_i$ are distinct, we can expand the initial condition in the eigenvector basis

$$\langle \boldsymbol{\pi}_0| = \alpha_1 \langle v_1| + \dots + \alpha_L \langle v_L|$$

  where $\alpha_i = \langle \boldsymbol{\pi}_0 | u_i \rangle$. This leads to

$$\langle \boldsymbol{\pi}_t| = \alpha_1 \langle v_1| e^{\lambda_1 t} + \dots + \alpha_L \langle v_L| e^{\lambda_L t} \tag{2.3}$$

# 2. Continuous-time Markov chains

- using (2.1) we have for $G = \big(g(x,y) : x,y \in S\big)$

$$p_{\Delta t}(x,y) = g(x,y)\Delta t + o(\Delta t) \quad \text{for all } x \neq y \in S\,.$$

So $g(x,y) \geq 0$ can be interpreted as transition rates.

$$p_{\Delta t}(x,x) = 1 + g(x,x)\Delta t + o(\Delta t) \quad \text{for all } x \in S\,,$$

and since $\sum_y p_{\Delta t}(x,y) = 1$ this implies that

$$g(x,x) = -\sum_{y \neq x} g(x,y) \leq 0 \quad \text{for all } x \in S\,.$$

- (2.2) can then be written intuitively as the **Master equation**

$$\frac{d}{dt}\pi_t(x) = \underbrace{\sum_{y \neq x} \pi_t(y) g(y,x)}_{\text{gain term}} - \underbrace{\sum_{y \neq x} \pi_t(x) g(x,y)}_{\text{loss term}} \quad \text{for all } x \in S\,.$$

- The Gershgorin theorem now implies that either $\lambda_i = 0$ or $\mathrm{Re}(\lambda_i) < 0$ for the eigenvalues of $G$, so there are no persistent oscillations.

# 2. Sample paths

Sample paths $t \mapsto X_t(\omega)$ are piecewise constant and right-continuous by convention.

For $X_0 = x$, define the **holding time** $\quad W_x := \inf\{t > 0 : X_t \neq x\}$ .

## Proposition 2.2

$W_x \sim \text{Exp}(|g(x,x)|)$, i.e. it is **exponentially distributed** with mean $1/|g(x,x)|$, and if $|g(x,x)| > 0$ the chain jumps to $y \neq x$ after time $W_x$ with probability $g(x,y)/|g(x,x)|$.

**Proof.** $W_x$ has the **memoryless property**, i.e. for all $t, u > 0$

$$\mathbb{P}(W_x > t+u | W_x > t) = \mathbb{P}(W_x > t+u | X_t = x) = \mathbb{P}(W_x > u)$$

where we used the Markov property and homogeneity. Therefore

$$\mathbb{P}(W_x > t+u) = \mathbb{P}(W_x > u)\mathbb{P}(W_x > t) \quad \Rightarrow \quad \mathbb{P}(W_x > t) = e^{\gamma t}$$

where $\quad \gamma = \dfrac{d}{dt}\mathbb{P}(W_x > t)\Big|_{t=0} = \lim\limits_{\Delta t \searrow 0} \dfrac{p_{\Delta t}(x,x) + o(\Delta t) - 1}{\Delta t} = g(x,x) \leq 0$ .

Conditioned on leaving the current state shortly, the probability to jump to $y$ is

$$\lim_{\Delta t \searrow 0} \frac{p_{\Delta t}(x,y)}{1 - p_{\Delta t}(x,x)} = \lim_{\Delta t \searrow 0} \frac{\Delta t\, g(x,y)}{1 - 1 - \Delta t\, g(x,x)} = \frac{g(x,y)}{-g(x,x)} \ .$$

# 2. Sample paths

- By the Markov property, subsequent holding times and jump probabilities are all independent.
- the **jump times** $J_0, J_1, \ldots$ are defined recursively as

$$J_0 = 0 \quad \text{and} \quad J_{n+1} = \inf\{t > J_n : X_t \neq X_{J_n}\} \, .$$

- The **jump chain** $(Y_n : n \in N_0)$ with $Y_n := X_{J_n}$
  is then a discrete-time Markov chain with transition matrix

$$p^Y(x,y) = \begin{cases} 0 & , \ x = y \\ g(x,y)/|g(x,x)| & , \ x \neq y \end{cases} \text{ if } g(x,x) > 0 \quad \text{and}$$
$$p^Y(x,y) = \delta_{x,y} \text{ if } g(x,x) = 0 \, .$$

- A sample path is constructed by simulating the jump chain $(Y_n : n \in \mathbb{N}_0)$ together with independent **holding times** $(W_{Y_n} : n \in \mathbb{N}_0)$, so that $J_n = \sum_{k=0}^{n-1} W_{Y_k}$

# 2. Examples

- A **Poisson process** with **rate** $\lambda$ (short PP($\lambda$)) is a CTMC with

$$S = \mathbb{N}_0,\ X_0 = 0 \quad \text{and} \quad g(x,y) = \lambda\delta_{x+1,y} - \lambda\delta_{x,y}\ .$$

  The PP($\lambda$) has **stationary and independent increments** with

$$\mathbb{P}[X_{t+u} = n+k | X_u = n] = p_t(0,k) = \frac{(\lambda t)^k}{k!}\, e^{-\lambda t} \quad \text{for all } u,t>0,\ k,n \in \mathbb{N}_0$$

  since $\pi_t(k) = p_t(0,k)$ solves the Master equation $\frac{d}{dt}\pi_t(k) = (\pi_t G)(k)$ .

- A **birth-death chain** with **birth rates** $\alpha_x$ and **death rates** $\beta_x$ is a CTMC with

$$S = \mathbb{N}_0 \quad \text{and} \quad g(x,y) = \alpha_x\delta_{x+1,y} + \beta_x\delta_{x-1,y} - (\alpha_x+\beta_x)\delta_{x,y}\ ,$$

  where $\beta_0 = 0$.

  Special cases include

  - $M/M/1$ server queues: $\alpha_x \equiv \alpha > 0$, $\beta_x \equiv \beta > 0$ for $x > 1$
  - $M/M/\infty$ server queues: $\alpha_x \equiv \alpha > 0$, $\beta_x = x\beta$
  - population growth model: $\alpha_x = x\alpha$, $\beta_x = x\beta$

# 2. Stationary distributions

## Definition 2.2

Let $(X_t : t \geq 0)$ be a homogeneous CTMC with state space $S$. The distribution $\pi(x),\ x \in S$ is called **stationary** if $\langle \boldsymbol{\pi} | G = \langle 0|$, or for all $y \in S$

$$\sum_{x \in S} \pi(x) g(x,y) = \sum_{x \neq y} \big( \pi(x) g(x,y) - \pi(y) g(y,x) \big) = 0 \, .$$

$\pi$ is called **reversible** if it fulfills the **detailed balance conditions**

$$\pi(x) g(x,y) = \pi(y) g(y,x) \quad \text{for all } x, y \in S \, .$$

- reversibility implies stationarity, since

$$\sum_{x \in S} \pi(x) g(x,y) = \sum_{x \in S} \pi(y) g(y,x) = 0 \, .$$

- Stationary distributions are left **eigenvectors** of $G$ with **eigenvalue** $0$ .

# 2. Stationary distributions

## Proposition 2.3 (Existence)

A DTMC or CTMC with finite state space $S$ has at least one stationary distribution.

**Proof.** Since $P$ and $G$ have row sum 1 and 0 we have $P|\mathbf{1}\rangle = |\mathbf{1}\rangle$ and $G|\mathbf{1}\rangle = |\mathbf{0}\rangle$
So 1 and 0 are eigenvalues, and left eigenvectors can be shown to have non-negative entries and thus can be normalized to be stationary distributions $\langle\boldsymbol{\pi}|$. □

**Remark.** If $S$ is countably infinite, stationary distributions may not exist, as for example for the SRW on $\mathbb{Z}$.

## Definition 2.3

A CTMC (or DTMC) is called **irreducible**, if for all $x, y \in S$

$$p_t(x,y) > 0 \text{ for some } t > 0 \qquad (p_n(x,y) > 0 \text{ for some } n \in \mathbb{N}) \ .$$

# 2. Stationary distributions

## Proposition 2.4 (Uniqueness)

An irreducible Markov chain has at most one stationary distribution.

**Proof.** Follows from the **Perron Frobenius theorem:**
Let $P$ be a stochastic matrix ($P = P_t$ for any $t \geq 0$ for CTMCs). Then

1. $\lambda_1 = 1$ is an eigenvalue of $P$, it is singular if and only if the chain is irreducible. Corresponding left and right eigenvectors have non-negative entries.
2. if the chain is continuous-time or discrete-time aperiodic, all remaining eigenvalues $\lambda_i \in \mathbb{C}$ satisfy $|\lambda_i| < 1$

The second part of the Perron Frobenius theorem also implies convergence of the transition functions to the stationary distribution, which is usually called ergodicity.

# 2. Ergodicity

## Theorem 2.5

An irreducible (aperiodic) MC with finite state space is **ergodic**, i.e. it has a unique stationary distribution $\boldsymbol{\pi}$ and

$$p_t(x,y) = \mathbb{P}[X_t = y | X_0 = x] \to \pi(y) \quad \text{as } t \to \infty\,, \quad \text{for all } x, y \in S\,.$$

## Theorem 2.6 (Ergodic Theorem)

Consider an ergodic Markov chain with unique stationary distribution $\boldsymbol{\pi}$. Then for every observable $f : S \to \mathbb{R}$ we have

$$\frac{1}{T}\int_0^T f(X_t)\,dt \quad \text{or} \quad \frac{1}{N}\sum_{n=1}^{N} f(X_n) \;\to\; \mathbb{E}_{\boldsymbol{\pi}}[f] \quad \text{as } T, N \to \infty\,.$$

- for a proof see e.g. [GS], chapter 9.5
- Stationary expectations can be approximated by time averages, which is the basis for a technique called **Markov chain Monte Carlo**.
- for example, choosing the indicator function $f = \mathbb{1}_x$ we get $\mathbb{E}_{\boldsymbol{\pi}}[f] = \pi(x)$

# 2. Reversibility

Consider a CTMC $(X_t : t \geq 0)$ which is **stationary**, i.e. $X_t \sim \boldsymbol{\pi}$ for $t \geq 0$. Then its definition can be extended to negative times to give a chain $(X_t : t \in \mathbb{R})$ .

## Proposition 2.7

Let $(X_t : t \geq \mathbb{R})$ be a stationary, finite state, irreducible CTMC with generator $G^X$. Then the **time reversed chain**

$$(Y_t : t \in \mathbb{R}) \quad \text{with} \quad Y_t := X_{-t}$$

is a stationary CTMC with generator $g^Y(x,y) = \dfrac{\pi(y)}{\pi(x)}\, g^X(y,x)$ .

- An analogous statement holds for stationary, finite state, irreducible DTMCs with $p^Y(x,y) = \dfrac{\pi(y)}{\pi(x)}\, p^X(y,x)$ .
- Stationary chains with reversible $\boldsymbol{\pi}$ are **time-reversible**, since $g^Y(x,y) = g^X(x,y)$ .
- **Example.** SRW on finite state space

# 2. Countably infinite state space

For infinite state space, Markov chains can get 'lost at infinity' and have no stationary distribution. Let $T_x := \inf\{t > J_1 : X_t = x\}$ be the first **return time** to a state $x$.

( For DTMCs return times are defined as $T_x := \inf\{n \geq 1 : X_n = x\}$ )

## Definition 2.4

A state $x \in S$ is called

- **transient**, if $\mathbb{P}[T_x = \infty | X_0 = x] > 0$
- **null recurrent**, if $\mathbb{P}[T_x < \infty | X_0 = x] = 1$ and $\mathbb{E}[T_x] = \infty$
- **positive recurrent**, if $\mathbb{P}[T_x < \infty | X_0 = x] = 1$ and $\mathbb{E}[T_x] < \infty$

- For an irreducible MC all states are either transient, null or positive recurrent. The MC has a unique stationary distribution if and only if it is positive recurrent.
- A transient CTMC can exhibit **explosion**. Define the **explosion time**

$$J_\infty := \lim_{n \to \infty} J_n \in (0, \infty] \quad \text{where } J_n \text{ are the jump times of the chain .}$$

The chain is called **non-explosive** if $\mathbb{P}[J_\infty = \infty] = 1$ .
This is always the case if $S$ is finite or $\sup_{x \in S} |g(x,x)| < \infty$ .

# 3. Interacting particle systems

- **'lattice'**: $\Lambda = \{1, \dots, L\}$ , finite set of points
- **state space** $S$ is given by the set of all **configurations**

$$\eta = \big(\eta(i) : i \in \Lambda\big) \in S = \{0,1\}^L \quad \text{(often also written } \{0,1\}^\Lambda\text{)} .$$

  $\eta(i) \in \{0,1\}$ signifies the presence of a particle/infection at site/individual $i$.
- Only local transitions are allowed with rates

$$\begin{aligned} \eta \to \eta^i \quad &\text{with rate} \quad c(\eta, \eta^i) \qquad \text{(reaction)} \\ \eta \to \eta^{ij} \quad &\text{with rate} \quad c(\eta, \eta^{ij}) \qquad \text{(transport)} \end{aligned}$$

where $\eta^i(k) = \begin{cases} \eta(k) & , \ k \neq i \\ 1 - \eta(k) & , \ k = i \end{cases}$ and $\eta^{ij}(k) = \begin{cases} \eta(k) & , \ k \neq i,j \\ \eta(j) & , \ k = i \\ \eta(i) & , \ k = j \end{cases}$

## Definition 3.1

An **interacting particle system (IPS)** is a CTMC with state space $S = \{0,1\}^\Lambda$ and generator $G$ with off-diagonal elements

$$g(\eta, \zeta) = \sum_{i \in \Lambda} \mathbb{1}_{\eta^i}(\zeta)\, c(\eta, \eta^i) \quad \Big( \text{or} \sum_{i,j \in \Lambda} \mathbb{1}_{\eta^{ij}}(\zeta)\, c(\eta, \eta^{ij}) \Big) .$$

# 3. Contact process

The contact process is a simple stochastic model for the **SI epidemic** with **infection rates** $q(i,j) \geq 0$ and uniform **recovery rate** 1.

## Definition 3.2

The **contact process (CP)** $(\eta_t : t \geq 0)$ is an IPS with rates

$$c(\eta, \eta^i) = \underbrace{\eta(i)}_{\text{recovery}} + \underbrace{\big(1-\eta(i)\big)\sum\nolimits_{j\neq i} q(j,i)\eta(j)}_{\text{infection}} \quad \text{for all } i \in \Lambda\,.$$

Usually, $q(i,j) = q(j,i) \in \{0, \lambda\}$, i.e. connected individuals infect each other with fixed rate $\lambda > 0$.

- The CP has one absorbing state $\eta(i) = 0$ for all $i \in \Lambda$, which can be reached from every initial configuration. Therefore the process is ergodic and the infection eventually gets **extinct** with probability 1.
- Let $T := \inf\{t > 0 : \eta_t \equiv 0\}$ be the **extinction time**. Then there exists a **critical value (epidemic threshold)** $\lambda_c > 0$ such that (for irreducible $q(i,j)$)

$$\mathbb{E}[T] \propto \log L \quad \text{for } \lambda < \lambda_c \quad \text{and} \quad \mathbb{E}[T] \propto e^{CL} \quad \text{for } \lambda > \lambda_c\,.$$

# 3. Voter model

The voter model describes opinion dynamics with **influence rates** $q(i,j) \geq 0$ at which individual $i$ persuades $j$ to switch to her/his opinion.

## Definition 3.3

The **linear voter model (VM)** $(\eta_t : t \geq 0)$ is an IPS with rates

$$c(\eta, \eta^i) = \sum_{j \neq i} \underbrace{q(j,i)\big(\eta(i)(1-\eta(j)) + (1-\eta(i))\eta(j)\big)}_{j \text{ influences } i \text{ if opinions differ}} \quad \text{for all } i \in \Lambda \, .$$

In non-linear versions the rates can be replaced by general (symmetric) functions.

- The VM is **symmetric** under relabelling opinions $0 \leftrightarrow 1$.
- If $q(i,j)$ is irreducible there are two absorbing states, $\eta \equiv 0, 1$, both of which can be reached from every initial condition. Therefore the VM is not ergodic, and **stationary measures** are

$$\alpha\delta_0 + (1-\alpha)\delta_1 \quad \text{with } \alpha \in [0,1] \text{ depending on the initial condition} \, .$$

- **Coexistence** of both opinions can occur on infinite lattices (e.g. $\mathbb{Z}^d$ for $d \geq 3$).

# 3. Exclusion process

The exclusion process describes transport of a conserved quantity (e.g. mass or energy) with **transport rates** $q(i,j) \geq 0$ site $i$ to $j$.

## Definition 3.4

The **exclusion process (EP)** $(\eta_t : t \geq 0)$ is an IPS with rates

$$c(\eta, \eta^{ij}) = q(i,j)\eta(i)\big(1 - \eta(j)\big) \quad \text{for all } i,j \in \Lambda\ .$$

The EP is called **simple (SEP)** if jumps occur only between nearest neighbours on $\Lambda$. The SEP is **symmetric (SSEP)** if $q(i,j) = q(j,i)$, otherwise **asymmetric (ASEP)**.

- The SEP is mostly studied in a 1D geometry with periodic or open boundaries.
- For periodic boundary conditions the total number of particles $N = \sum_i \eta(i)$ is **conserved**. The process is ergodic on the sub-state space

$$S_N = \left\{\eta \in \{0,1\}^L : \sum_i \eta(i) = N\right\}$$

  for each value $N = 0, \ldots L$, and has a unique stationary distribution.
- For open boundaries particles can be created and destroyed at the boundary, the system is ergodic on $S$ and has a unique stationary distribution.

# 4. Graphs - definition

## Definition 4.1

A **graph** (or **network**) $G = (V, E)$ consists of a finite set $V = \{1, \ldots, N\}$ of **vertices** (or **nodes**, **points**), and a set $E \subseteq V \times V$ of **edges** (or **links**, **lines**).
The graph is called **undirected** if $(i, j) \in E$ implies $(j, i) \in E$, otherwise **directed**.
The structure of the graph is encoded in the **adjacency** (or **connectivity**) **matrix**

$$A = (a_{ij} : i, j \in V) \quad \text{where} \quad a_{ij} = \begin{cases} 1\,, & (i,j) \in E \\ 0\,, & (i,j) \notin E \end{cases}.$$

We denote the number of edges by $K = |E|$ for directed, or $K = |E|/2$ for undirected graphs.

- Graphs we consider do not have self edges, i.e. $(i, i) \notin E$ for all $i \in V$, or multiple edges, since edges $(i, j)$ are unique elements of $E$.
- **Weighted graphs** with edge weights $w_{ij} \in \mathbb{R}$ can be used to represent continuous- or discrete-time Markov chains.
- In general graphs can also infinite, but we will focus on finite graphs. Many of the following graph characteristics only make sense in the finite case.

# 4. Graphs - paths and connectivity

## Definition 4.2

A **path** $\gamma_{ij}$ of length $l = |\gamma_{ij}|$ from vertex $i$ to $j$ is sequence of vertices

$$\gamma_{ij} = (v_1 = i, v_2, \ldots, v_{l+1} = j) \quad \text{with} \quad (v_k, v_{k+1}) \in E \text{ for all } k = 1, \ldots, l \,,$$

and $v_k \neq v_{k'}$ for all $k \neq k' \in \{1, \ldots, l\}$ (i.e. each vertex is visited only once).
If such a path exists, we say that vertex $i$ is **connected** to $j$ (write $i \to j$).
**Shortest paths** between vertices $i, j$ are called **geodesics** (not necessarily unique) and their length $d_{ij}$ is called the **distance** from $i$ to $j$. If $i \not\to j$ we set $d_{ij} = \infty$.
A graph is **connected** if $d_{ij} < \infty$ for all $i, j \in V$.
The **diameter** and the **characteristic path length** of the graph $G$ are given by

$$\operatorname{diam}(G) := \max\{d_{ij} : i, j \in V\} \in \mathbb{N}_0 \cup \{\infty\} \,,$$

$$L = L(G) := \frac{1}{N(N-1)} \sum_{i,j \in V} d_{ij} \in [0, \infty] \,.$$

For undirected graphs we have $d_{ij} = d_{ji}$ which is finite if $i \leftrightarrow j$, and they can be decomposed into **connected components**, where we write

$$C_i = \{j \in V : j \leftrightarrow i\} \quad \text{for the component containing vertex } i \,.$$

# 4. Graphs - degrees

## Definition 4.3

The **in-** and **out-degree** of a node $i \in V$ is defined as

$$k_i^{\text{in}} = \sum_{j \in V} a_{ji} \quad \text{and} \quad k_i^{\text{out}} = \sum_{j \in V} a_{ij} \ .$$

$k_1^{\text{in}}, \ldots k_N^{\text{in}}$ is called the **in-degree sequence** and the **in-degree distribution** is

$$\left(p^{\text{in}}(k) : k \in \{0, \ldots, K\}\right) \quad \text{with} \quad p^{\text{in}}(k) = \tfrac{1}{N} \sum_{i \in V} \delta_{k, k_i^{\text{in}}}$$

giving the fraction of vertices with in-degree $k$. The same holds for out-degrees, and in undirected networks we simply write $k_i = k_i^{\text{in}} = k_i^{\text{out}}$ and $p(k)$.

- Note that $\sum_{i \in V} k_i = \sum_{i,j \in V} a_{ij} = |E|$ (also for directed), average and variance are

$$\langle k \rangle = \tfrac{1}{N} \sum_{i \in V} k_i = |E|/N = \sum_k k p(k) \ , \quad \sigma^2 = \langle k^2 \rangle - \langle k \rangle^2 \ .$$

- In a **regular graph** (usually undirected) all vertices have equal degree $k_i \equiv k$.
- Graphs where the degree distribution shows a power law decay, i.e. $p(k) \propto k^{-\alpha}$ for large $k$, are often called **scale-free**.
  Real-world networks are often scale-free with exponent around $\alpha \approx 3$.

# 4. Graphs - first examples

## Example 2 (Some graphs)

The **complete graph** $K_N$ with $N$ vertices is an undirected graph where all $N(N-1)/2$ vertices $E = ((i,j) : i \neq j \in V)$ are present.
**Regular lattices** $\mathbb{Z}^d$ with edges between nearest neighbours are examples of regular graphs with degree $k = 2d$.

## Definition 4.4

A **tree** is an undirected graph where any two vertices are connected by exactly one path. Vertices with degree 1 are called **leaves**.
In a **rooted tree** one vertex $i \in V$ is the designated **root**, and the graph can be directed, where all vertices point towards or away from the root.

A **cycle** is a closed path $\gamma_{ii}$ of length $|\gamma_{ii}| > 2$. $G$ is a tree if and only if

- it is connected and has no cycles;
- it is connected but is not connected if a single edge is removed;
- it has no cycles but a cycle is formed if any edge is added.

# 4. Graphs - degree correlations

## Definition 4.5

For undirected graphs, the **joint degree distribution** of nodes linked by an edge is

$$q(k,k') = \frac{1}{|E|}\sum_{(i,j)\in E}\delta_{k_i,k}\delta_{k_j,k'} = \frac{\sum_{i,j\in V} a_{ij}\delta_{k_i,k}\delta_{k_j,k'}}{\sum_{i,j\in V} a_{ij}} = q(k',k)\ .$$

With the marginal $q(k') = \sum_k q(k,k')$ we have the **conditional degree distribution**

$$q(k|k') = q(k,k')/q(k') \quad \text{with average} \quad k_{nn}(k') := \sum_k kq(k|k')\ .$$

The network is called **uncorrelated** if $k_{nn}(k')$ is independent of $k'$, **assortative** if $k_{nn}(k') \nearrow$ in $k'$ and **disassortative** if $k_{nn}(k') \searrow$ in $k'$.

- The marginal $q(k)$ corresponds to **edge biased degree sampling**, i.e.

$$q(k) = \sum_{k'} q(k,k') = \frac{1}{|E|}\sum_{i,j\in V} a_{ij}\delta_{k_i,k} = \frac{N}{|E|}\frac{1}{N}\sum_{i\in V} k_i\delta_{k_i,k} = \frac{kp(k)}{\langle k\rangle}\ .$$

  For uncorrelated networks $q(k|k') = q(k)$ and thus $\quad k_{nn}(k') = \langle k^2\rangle/\langle k\rangle$ .

# 4. Subgraphs

- The degree of correlation can be quantified by the **correlation coefficient**

$$\chi := \frac{\langle kk' \rangle_q - \langle k \rangle_q^2}{\langle k^2 \rangle_q - \langle k \rangle_q^2} = \frac{\sum_{k,k'} kk' \left( q(k,k') - q(k)q(k') \right)}{\sum_k k^2 q(k) - \left( \sum_k kq(k) \right)^2} \in [-1,1] \, .$$

## Definition 4.6

A **subgraph** $G' = (V', E')$ of $G = (V, E)$ is a graph such that $V' \subseteq V$ and $E' \subseteq E$.

- Small connected subgraphs are also called **motifs**, the simplest non-trivial examples in undirected graphs are connected triples and triangles.
- Fully connected (complete) subgraphs which are maximal with respect to connectedness are called **cliques**.
- A **spanning tree** is a tree subgraph that contains all vertices of the graph.
- A subgraph $G'$ is called a **community**, if (for example)

$$\sum_{i,j \in V'} a_{ij} > \sum_{i \in V', j \notin V'} a_{ij} \quad \text{(there are also other definitions)} \, .$$

# 4. Clustering

Clustering aims to quantify the probability that two neighbours of a given vertex are themselves neighbours. Two different definitions are used in the literature.

## Definition 4.7

The **global clustering coefficient** for an undirected graph is defined as

$$C = \frac{3 \times \#\text{ of (connected) triangles}}{\#\text{ of (connected) triples}} = \frac{3\sum_{i<j<l} a_{ij}a_{jl}a_{li}}{\sum_{i<j<l}(a_{ij}a_{il} + a_{ji}a_{jl} + a_{li}a_{lj})} \in [0,1] \,.$$

Alternatively, one can define a **local clustering coefficient**

$$C_i = \frac{\#\text{ of triangles containing vertex } i}{\#\text{ of triples centered on vertex } i} = \frac{\sum_{j<l} a_{ij}a_{jl}a_{li}}{\sum_{j<l} a_{ij}a_{il}} \in [0,1] \,,$$

and use the average $\langle C_i \rangle = \frac{1}{N}\sum_i C_i$ to quantify clustering.

- For a tree we have $C = \langle C_i \rangle = 0$ and for the complete graph $C = \langle C_i \rangle = 1$ .
- Higher-order clustering coefficients can be defined similarly, using different subgraphs as basis.

# 4. Graph spectra

## Definition 4.8

The **spectral density** of a graph $G = (V, E)$ is

$$\rho(\lambda) := \tfrac{1}{N} \sum_{i \in V} \delta(\lambda - \lambda_i) \quad \text{where} \quad \lambda_1, \ldots, \lambda_N \in \mathbb{C}$$

are the eigenvalues of the adjacency matrix $A$.

- **Perron-Frobenius:** $A$ has a real eigenvalue $\lambda_1$ with maximal modulus and real, non-negative eigenvector(s), and the **multiplicity of $\lambda_1$ equals the number of connected components** in undirected graphs. Properly chosen orthogonal eigenvectors to $\lambda_1$ have non-zero entries on the individual connected components.
- $(A^n)_{ij}$ is equal to the **number of walks** (paths which allow repeated vertices) from $i$ to $j$ of length $n$. We also have

  $$\mathrm{Tr}(A^n) = \sum_{i=1}^{N} \lambda_i^n \quad \text{and} \quad \big(\mathrm{Tr}(A)\big)^n = 0 \,,$$

  which can be used to derive statements like:

  $$\sum_{i<j} \lambda_i \lambda_j = -|E| \,, \quad \sum_{i<j<l} \lambda_i \lambda_j \lambda_l = 2 \cdot \#\text{ of triangles in } G \,.$$

# 4. Graph Laplacian

## Definition 4.9

The **Graph Laplacian** for a graph $(V, E)$ with adjacency matrix $A$ is defined as

$$Q := A - D \quad \text{where} \quad D = \left( \delta_{ij} \sum_{l \neq i} a_{il} : i, j \in V \right).$$

- $Q$ has eigenvalues in $\mathbb{C}$ with real part $\mathrm{Re}(\lambda) < 0$ except for $\lambda_1 = 0$, which follows directly from the Gershgorin theorem and vanishing row sums. Again, multiplicity of $\lambda_1 = 0$ equals the number of connected components in $G$. The smaller the second largest real part of an eigenvalue, the harder it is to cut $G$ into separated components by removing edges.
- $Q$ defines a generator matrix of a continuous-time random walk on $V$ with transition rates $a_{ij}$. Using weighted graphs, any finite state CTMC can be represented in this way.
- The Laplacian determines the first order linearized dynamics of many complex processes on graphs and is therefore of particular importance.

# 4. More general graphs and networks

- For **multigraphs**, multiple edges between nodes and loops ($a_{ii} > 0$) are allowed.
- **Hypergraphs** $(V, E)$ are generalizations in which an edge can connect any number of vertices. Formally, the set of **hyperedges** $E \subseteq \mathcal{P}(V)$ is a set of non-empty subsets of $V$.
- In **bipartite graphs** the edge set can be partitioned into two sets $V_1, V_2 \subseteq V$ each non-empty, with no connections within themselves, i.e. $a_{ij} = a_{ji} = 0$ for all $i, j \in V_1$ and also for all $i, j \in V_2$.
  Simple undirected examples include regular lattices $\mathbb{Z}^d$ for $d \geq 1$ which are partitioned into sites with even and odd parity. Feed-forward neural networks are examples of directed graphs with bipartite or multi-partite structure.
- **Multilayer networks** $M = (\mathbf{G}, \mathbf{C})$ consist of a family of $m$ (weighted or unweighted) graphs $G_\alpha = (V_\alpha, E_\alpha)$ (called **layers** of $M$), and the set of interconnections between nodes of different layers

$$\mathbf{C} = \left\{ c_{\alpha,\beta} \subseteq V_\alpha \times V_\beta : \alpha, \beta \in \{1, \ldots, m\},\ \alpha \neq \beta \right\} .$$

  Real examples include transportation networks or social networks with different types of connections.

# 5. E-R Random graphs

## Definition 5.1

An **(Erdős-Rényi) random graph** $G \sim \mathcal{G}_{N,K}$ has uniform distribution on the set of all undirected graphs with $N$ vertices and $K = |E|/2$ edges, i.e.

$$\mathbb{P}[G = (V, E)] = \delta_{|V|,N}\delta_{|E|/2,K} \Big/ \binom{N(N-1)/2}{K} .$$

An **(E-R) random graph** $G \sim \mathcal{G}_{N,p}$ has $N$ vertices and each (undirected) edge is present independently with probability $p \in [0, 1]$, i.e.

$$\mathbb{P}[G = (V, E)] = \delta_{|V|,N} p^{|E|/2}(1-p)^{N(N-1)/2-|E|/2} .$$

- The ensemble $\mathcal{G}_{N,p}$ is easier to work with and is mostly used in practice, and for $N, K$ large, $\mathcal{G}_{N,K}$ is largely equivalent to $\mathcal{G}_{N,p}$ with $p = K/N$.
- Since edges are present independently, graphs $G \in \mathcal{G}_{N,p}$ should typically be uncorrelated. Indeed, one can show that $\chi(G)$, $\mathbb{E}[\chi] \to 0$ as $N \to \infty$.

# 5. E-R Random graphs - properties

- The number of undirected edges for $G \sim \mathcal{G}_{N,p}$ is **random**, $K \sim \mathrm{Bi}\left(\frac{N(N-1)}{2}, p\right)$ .
  Thus, the average degree is **random**, $\langle k \rangle = 2K/N$ with $\mathbb{E}[\langle k \rangle] = (N-1)p$.
- The expected number of triangles in a $\mathcal{G}_{N,p}$ graph is $\binom{N}{3}p^3$ ,
  and the number of triples is $\binom{N}{3}3p^2$ .
  Since fluctuations are of lower order, this implies for all $G_N \sim \mathcal{G}_{N,p}$

$$C(G_N) = \frac{3\binom{N}{3}p^3(1+o(1))}{\binom{N}{3}3p^2(1+o(1))} \to 3p^3/(3p^2) = p \quad \text{as } N \to \infty \, .$$

- The **expected** degree distribution for $G_N \sim \mathcal{G}_{N,p}$ is binomial. In the limit $N \to \infty$ with $p = p_N = z/(N-1)$ keeping $z = \mathbb{E}[\langle k \rangle]$ fixed we have

$$\mathbb{E}[p(k)] = \mathbb{P}[k_i = k] = \binom{N-1}{k} p_N^k (1-p_N)^{N-1-k} \to \frac{z^k}{k!} e^{-z} \, .$$

  Therefore, E-R $\mathcal{G}_{N,p}$ graphs are sometimes called **Poisson random graphs**.
- In this scaling limit E-R graphs are **locally tree-like**, i.e. finite connected components

$$C_i^n := \{j \in V : j \leftrightarrow i, d_{ij} \leq n\} \, , \quad n \text{ fixed}$$

  are tree subgraphs as $N \to \infty$ with probability 1.
  Vertex degrees are $k_i \sim \mathrm{Poi}(z)$ and iid $k_j \sim \mathrm{Poi}(z) + 1$.

# 5. Percolation

Percolation studies robustness of connectivity properties of graphs under deletion of edges or vertices (e.g. random attacks or immunization).

## Definition 5.2

Consider a connected, undirected graph $G = (V, E)$. **Bond percolation** is a static probabilistic model with state space

$$S = \Omega = \{0,1\}^E = \left\{ e_{ij} \in \{0,1\} : (i,j) \in E \right\} ,$$

and distribution $p = \mathbb{P}[e_{ij} = 1] = 1 - \mathbb{P}[e_{ij} = 0]$, i.e. $e_{ij} \sim \mathrm{Be}(p)$ iid with $p \in [0,1]$. Edges $(i,j) \in E$ are called **open** if $e_{ij} = 1$ and **closed** if $e_{ij} = 0$, and we denote by

$$G^o = (V, E^o) \quad \text{with} \quad E^o = \{(i,j) \in E : e_{ij} = 1\} \subseteq E$$

the (random) subgraph containing only open edges. A sequence of connected graphs $G_N$ of increasing size $|V_N| = N$ exhibits **percolation with parameter** $p$ if

$$|\bar{C}_N^o|/N \geq c > 0 \quad \text{as } N \to \infty \quad \text{with probability 1} ,$$

where $|\bar{C}_N^o| = \max_{i \in V_N} |C_i^o|$ is the size of the largest connected component $\bar{C}_N^o$ of $G_N^o$.

# 5. Percolation and E-R graphs

- Alternatively, percolation can be defined on an infinite graph $G$ (e.g. $\mathbb{Z}^d$) with

$$\textbf{percolation probability} \quad \theta(p) = \mathbb{P}[|C_0| = \infty] \begin{cases} = 0\,, & \text{for } p < p_c \\ > 0\,, & \text{for } p > p_c \end{cases},$$

  changing behaviour at a **critical value** $p_c \in [0,1]$.
- In **site percolation** vertices and their adjacent edges are deleted.
- E-R random graphs $\mathcal{G}_{N,p}$ have the same distribution as open subgraphs $G_N^o \subseteq K_N$ under bond percolation with parameter $p$ on the complete graph $K_N$.

## Theorem 5.1 (Giant component for E-R graphs)

Consider $G_{N,p} \sim \mathcal{G}_{N,p}$ with $p = z/N$ and maximal connected component $\bar{C}_{N,p}$. Then

$$|\bar{C}_{N,p}| = \begin{cases} O(\log N)\,, & \text{for } z < 1 \\ O(N^{2/3})\,, & \text{for } z = 1 \\ O(N)\,, & \text{for } z > 1 \end{cases}.$$

$c(z) := \lim_{N\to\infty} |\bar{C}_{N,p}|/N$ is a continuous function of $z$. For $z > 1$, $\bar{C}_{N,p}$ is the only **giant component** of size $O(N)$, and the second largest is of order $O(\log N)$.

Local trees with $1 + \text{Poi}(z)$ degrees die out with probability 1 if and only if $z \leq 1$.

# 5. The Wigner semi-circle law

## Theorem 5.2 (Wigner semi-circle law)

Let $A = (a_{ij} : i,j = 1,\ldots,N)$ be a real, symmetric matrix with iid entries $a_{ij}$ for $i \leq j$ with finite moments, and $\mathbb{E}[a_{ij}] = 0$, $\mathrm{var}[a_{ij}] = \sigma^2$ (called a **Wigner matrix**). Then the spectral density $\rho_N$ of the matrix $A/\sqrt{N}$ converges with probability one to

$$\rho_N(\lambda) \to \rho_{sc}(\lambda) := \begin{cases} (2\pi\sigma^2)^{-1}\sqrt{4\sigma^2 - \lambda^2}\,, & \text{if } |\lambda| < 2\sigma \\ 0 \qquad\qquad\quad\;\; , & \text{otherwise} \end{cases}.$$

- The bulk of eigenvalues of unscaled Wigner matrices typically lies in the interval $[-2\sqrt{N}\sigma, 2\sqrt{N}\sigma]$.
- Adjacency matrices $A$ of $\mathcal{G}_{N,p}$ random graphs are symmetric with iid $\mathrm{Be}(p)$ entries with $\mathbb{E}[a_{ij}] = p$ and $\mathrm{var}[a_{ij}] = p(1-p)$, so are not Wigner matrices.
  $A$ has a maximal **Perron-Frobenius eigenvalue** of order $pN$, but all other eigenvalues have modulus of order $\sqrt{N}$.
  For fixed $p > 0$ the Wigner semi-cirlce law holds for $N \to \infty$ as stated above.
  For scaled $p = p_N \gg p_c = 1/N$ the width of the support reduces to $4\sqrt{N}\sigma_N$ with $\sigma_N = \sqrt{p_N}$ and a modified version holds.
  For $p = p_N \ll p_c = 1/N$ the asymptotic spectral density deviates from $\rho_{sc}$.

# 5. Preferential attachment

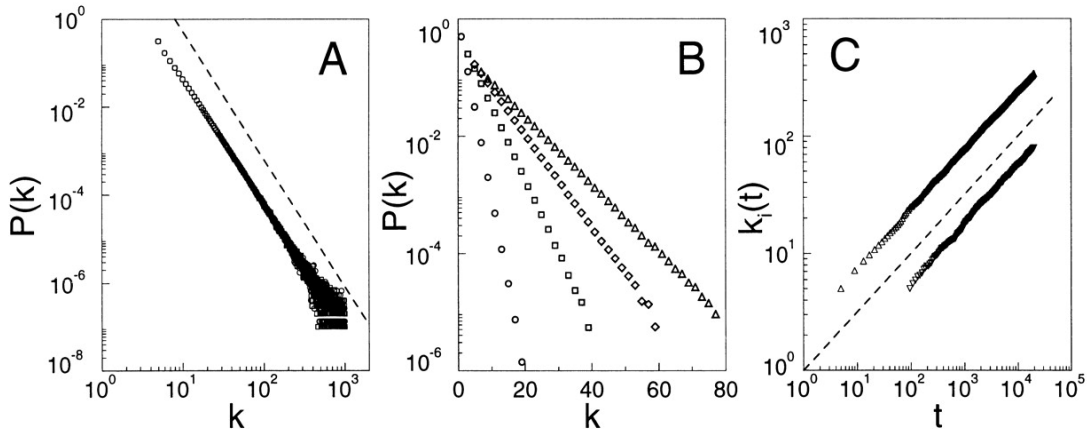
The prevalence of power-law degree distributions in real complex networks can be attributed to growth mechanisms subject to **preferential attachment**.

## Definition 5.3

Starting with a complete graph $(V_0, E_0)$ of $|V_0| = m_0$ nodes, at each time step $t = 1, \ldots, N - m_0$ a new node $j = t + m_0$ is added. It forms $m \leq m_0$ undirected edges with existing nodes $i \in V_{t-1}$ with a probability proportional to their degree $\pi_{j \leftrightarrow i} = k_i / \sum_{l \in V_t} k_l$ (preferential attachment).
The resulting, undirected graph with $N$ nodes and $K = m_0(m_0 - 1)/2 + m(N - m_0)$ is called a **Barabási-Albert** random graph, denoted by $\mathcal{G}^{\mathrm{BA}}_{N,K}$.

- As $N \to \infty$, the average degree is $\langle k \rangle = 2m$ and the degree distribution $p_N(k)$ converges to a distribution $p(k)$ with **power law tail**, i.e. $p(k) = Ck^{-\alpha}$ as $k \to \infty$ where $\alpha = 3$, which is close to exponents observed for real-world networks. This is independent of the parameters $m_0$ and $m$.
- Characteristic path length and clustering coefficient typically behave like $L = O(\log N)$ and $C = O(N^{-0.75})$ for $\mathcal{G}^{\mathrm{BA}}_{N,K}$ graphs.
- The average degree of nodes increases with their age.

# 5. Preferential attachment




(A) $m_0 = m = 5, N = 150K,\ 200K$, (B) $\gamma = 0, m_0 = m = 1,\ 3,\ 5,\ 7$, (C) $t_1 = 5,\ t_2 = 95$

taken from [A.-L. Barabási, R. Albert, Science **286**(5439), 509-512 (1999)]

- Variations of the model connecting to vertices $i$ with probability proportional to $k_i + k_0$ lead to power law degree distributions with $\alpha = 3 + k_0/m$.
- For **non-linear preferentail attachment** proportional to $k_i^\gamma$ the degree distribution converges to a stretched exponential tail $\exp(-Ck^{1-\gamma})$ for $\gamma \in [0, 1)$, and for $\gamma > 1$ a single vertex connects to almost all other vertices.
- There are many other and also older models that develop the idea of preferential attachment, for some of which one can do exact computations.
  (see e.g. [M.E.J. Newman, Siam Review **45**(2), 167256 (2003)]) .

# 5. Small-world networks

## Definition 5.4

A sequence of connected graphs $G_N$ with increasing size $|V_N| = N$ exhibits the **small-world property**, if the characteristic path length $L(G_N) = O(\log N)$.

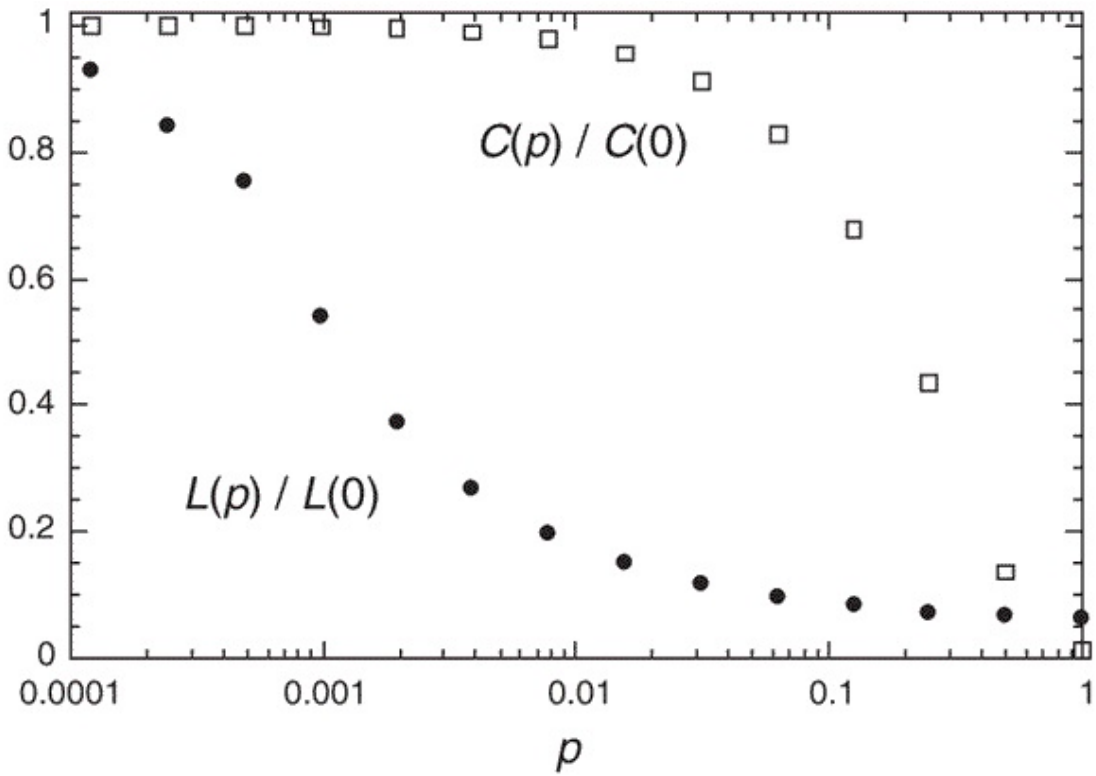
Examples include trees with degrees $k_i \leq 3$ and also the giant or largest component in E-R random graphs. In most graph models small-worldness is paired with low clustering coefficients, e.g. 0 for trees and $p$ for $\mathcal{G}_{N,p}$ graphs. However, many real examples of small world networks exhibit also large clustering coefficients, such as networks of social contacts.

## Definition 5.5

Consider a $2m$-regular ring graph with adjacency matrix $a_{ij} = \begin{cases} 1\,, & |i-j| \leq m \\ 0\,, & \text{otherwise} \end{cases}$ of size $N$ with a total number of $K = mN$ undirected edges.
For all $i$, each link $(i,j)$ to a clockwise neighbour with $j > i$ is **rewired** with probability $p \in [0,1]$, i.e. replaced by a link $(i,l)$ where $l$ is chosen uniformly among vertices not adjacent to $i$. The resulting graph is a **Watts-Strogatz random graph**, denoted by $\mathcal{G}^{\mathrm{WS}}_{N,K}$.

# 5. Watts-Strogatz model

- W-S random graphs interpolate between a regular lattice for $p = 0$ and a $\mathcal{G}_{N,K}$ E-R random graph conditioned on the event that all vertices have degree $k_i \geq m$.
- Expected clustering coefficient $\mathbb{E}[C(p)]$ and characteristic path length $\mathbb{E}[L(p)]$ are monotone decreasing functions of $p$ and show the following behaviour.




$N = 1000$ and $m = 5$, taken from [D.J. Watts, S.H. Strogatz, Nature **393**, 440-442 (1998)]

# 5. Configuration model

## Definition 5.6

The **configuration model** $\mathcal{G}_{N,D}^{\text{conf}}$ is defined as the uniform distribution among all undirected graphs with $N$ vertices with a given degree sequence $D = \{k_1, \ldots, k_N\}$, such that $\sum_{i \in V} k_i = 2K$.

- Not all sequences $D$ that sum to an even number are **graphical**.
- Sampling is usually done by attaching $k_i$ half edges to each vertex $i$ and matching them randomly. This can lead to self loops and rejections.
- General **randomized graphs** with given degree distribution $p(k)$ can be sampled in the same way. If $k_{\max} = \max_i k_i$ is bounded, one can show that these graphs exhibit a giant (connected) component of size $O(N)$ if

$$Q := \sum_{k \geq 0} k(k-2)p(k) > 0 \,,$$

  and if $Q < 0$ the largest component is of size $O(k_{\max}^2 \log N)$.
- For directed versions with $D^{\text{in}}$ and $D^{\text{out}}$ we need $\sum_{i \in V} k_i^{\text{in}} = \sum_{i \in V} k_i^{\text{out}}$.

# 6. Planar graphs and spatial point processes

## Definition 6.1

A **planar graph** is an undirected graph that can be embedded in the plane, i.e. it can be drawn in such a way that no edges cross each other. The edges of a particular embedding partition the plane into **faces**. A connected planar graph $G$ has a **dual graph** $G^*$, which has one vertex in each face of $G$, and a unique edge crossing each edge of $G$. $G^*$ may be a multigraph with self-loops.
A **maximal planar graph** is called a **triangulation**.

- Every planar graph is 4-partite or 4-colourable.
- In a triangulation each face is bounded by three edges. By induction, every triangulation with $N>2$ nodes has $K=3N-6$ undirected edges and $2N-4$ faces.

## Definition 6.2

A random countable set $\Pi \subseteq \mathbb{R}^d$ is called a **spatial point process**.
$\Pi \subseteq \mathbb{R}^d$ is called a homogeneous **Poisson point process** $\mathrm{PPP}(\lambda)$ with rate $\lambda > 0$ if

- for all $A \subseteq \mathbb{R}^d$ we have $N(A) := |\Pi \cap A| \sim \mathrm{Poi}(\lambda|A|)$ ,
- for all disjoint $A_1, \ldots, A_n \subseteq \mathbb{R}^d$, $N(A_1), \ldots, N(A_n)$ are independent.

# 6. Planar graphs and spatial point processes

- To sample from a PPP($\lambda$) e.g. in a box $A = [0, L]^d$, pick $N(A) \sim \text{Poi}(\lambda L^d)$, then place $N(A)$ particles independently in $A$ each with uniform distribution, i.e. pick the $d$ coordinates uniformly in $[0, L]$.
- A Poisson process PP($\lambda$) is equivalent to a PPP($\lambda$) on $[0, \infty)$.

## Definition 6.3

Let $\Pi = \{x_1, x_2, \ldots\}$ be a countable subset of $\mathbb{R}^d$, endowed with a distance function $d(x, y)$. A **Voronoi tesselation (or diagram)** is given by the family of **Voronoi cells** $\{A_1, A_2, \ldots\} \subseteq \mathbb{R}^d$ where

$$A_i = \left\{x \in \mathbb{R}^d : d(x, x_i) \leq d(x, x_j) \text{ for all } j \neq i\right\}$$

is the set of points closest to $x_i$.

**Properties in 2 dimensions.**

- The shape of Voronoi cells depends on the distance function, for Euclidean distance $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ they are convex polygons, and boundaries between adjacent cells are straight lines.
- The dual graph of a Voronoi diagram of a set $\Pi$ is called **Delaunay triangulation**, which is not unique if 4 or more cells intersect in a point.