Networks and Random Processes

Class test

The class test counts 25/100 module marks. [x] indicates weight of each question.
Attempt all 5 questions.

1. (a) State the weak law of large numbers and the central limit theorem.
   (b) Define the Erdős-Rényi random graph model $\mathcal{G}_{N,p}$, including the set of all possible graphs and the corresponding probability distribution.
   Compute the expected degree distribution.
   (c) Define what it means for a real-valued process $(M_t : t \geq 0)$ to be a martingale.
   State Itô's formula for a process $(X_t : t \geq 0)$ on state space S with generator $\mathcal{L}$ and a function $f : S \to \mathbb{R}$ which does not explicitly depend on time. Include the expression for the quadratic variation of the martingale.
   (d) Give the generator of the Poisson process $(N_t : t \geq 0)$ with rate $\lambda > 0$.
   Use Itô's formula to show that $N_t - \lambda t$ is a martingale and compute its quadratic variation. [20]

2. Consider the undirected graph G with adjacency matrix $A = \begin{pmatrix} 0&1&1&1&0&0 \\ 1&0&1&0&1&0 \\ 1&1&0&0&0&1 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \end{pmatrix}$.
   (a) Draw the graph G. Identify a clique of vertices and draw a spanning tree of G.
   (b) Give the matrix of vertex distances d_{ij} and compute the characteristic path length $L(G)$ and the diameter $\mathrm{diam}(G)$ of G.
   (c) Give the degree sequence $(k_1, \ldots, k_6)$ and compute the degree distribution $p(k)$ and the average degree $\langle k \rangle$ of G.
   (d) Compute the global clustering coefficient C and the average $\langle C_i \rangle$ of the local clustering coefficients C_i.
   (e) Give all non-zero entries of the joint degree distribution $q(k, k')$.
   Compute the marginal $q(k)$. For all k' with $q(k') > 0$ compute the conditional distribution $q(k|k')$ and the corresponding expectation $k_{nn}(k')$. [20]

3. (a) State two equivalent definitions of standard Brownian motion. See notes
   (b) Let $(B_t : t \geq 0)$ be a standard Brownian motion. Prove that for any $\lambda > 0$, the process $(X_t : t \geq 0)$ with $X_t := \frac{1}{\lambda} B_{t\lambda^2}$ is also a standard Brownian motion.
   (c) State the definition of a diffusion process on $\mathbb{R}$. See notes
   From now on, consider the Ornstein-Uhlenbeck process $(X_t : t \geq 0)$ given by the SDE
   $$dX_t = -\alpha X_t dt + \sigma dB_t \quad \text{with } \alpha > 0 \quad \text{and} \quad X_0 = x_0 \text{ (deterministic)}.$$
   (d) Write down the generator of this process.
   Derive equations for the mean $m(t) := \mathbb{E}[X_t]$ and the variance $v(t) := \mathbb{E}[X_t^2] - \mu(t)^2$ and solve them with the above deterministic initial condition $X_0 = x_0$.
   (e) Is $(X_t : t \geq 0)$ a Gaussian process? Yes $X_t \sim N(m(t), v(t))$
   Use the result of (d) to specify the distribution of X_t for all $t \geq 0$, and also give the stationary distribution as $t \to \infty$. $\to N(0, \sigma^2/2\alpha)$ wst... [20]

d) $\mathcal{L}f(x) = -\alpha x f'(x) + \frac{\sigma^2}{2} f''(x)$
$\mathcal{L}x = -\alpha x \Rightarrow \dot{m}(t) = -\alpha m(t) \Rightarrow m(t) = x_0 e^{-\alpha t}$
... $e^{-2\alpha t}$ ... $\Rightarrow v(t) = m(t) - m(t)^2 = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})$

1.a) Let $X_1, X_2, \ldots$ be iid rv's with $\mu := \mathbb{E}[X_i] < \infty$ and $\mathbb{E}[|X_i|] < \infty$.
Then $\frac{1}{n}\sum_{i=1}^n X_i \to \mu$ in distribution as $n \to \infty$.
X_i as above with $\sigma^2 := \mathrm{Var}(X_i) < \infty$. Then $\frac{1}{\sqrt{n}}\sum_{i=1}^n (X_i - \mu) \to \xi \sim N(0, \sigma^2)$ in distr.

b) For $\mathcal{G}_{N,p}$: $\mathbb{P}[G = (V, E)] = \delta_{|V|,N}\, p^{|E|/2} (1-p)^{N(N-1)/2 - |E|/2}$
$\mathbb{E}[p(k)] = \mathbb{E}[\frac{1}{N}\sum_{i=1}^N \delta_{k_i,k}] = \mathbb{P}[k_i = k] = \binom{N-1}{k} p^k (1-p)^{N-1-k}$

c) $(M_t : t \geq 0)$ is a martingale w.r.t. $(X_t : t \geq 0)$ if $\mathbb{E}[|M_t|] < \infty$ $\forall t \geq 0$ and
$\mathbb{E}[M_t | (X_u : 0 \leq u \leq s)] = M_s$ for all $0 \leq s < t$
Itô: Let $(X_t : t \geq 0)$ be a MP with generator $\mathcal{L}$, then for $f : S \to \mathbb{R}$
$f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\, ds$ is a martingale w.r.t. $(X_t : t \geq 0)$
with quadr. variation $[M]_t = \int_0^t [\mathcal{L}f^2(X_s) - 2f(X_s)\mathcal{L}f(X_s)]\, ds$

d) $\mathcal{L}f(n) = \lambda[f(n+1) - f(n)] \Rightarrow \mathcal{L}n = \lambda \Rightarrow N_t - \underbrace{N_0}_{=0} - \int_0^t \lambda\, ds = N_t - \lambda t$ is m'gale
$\mathcal{L}n^2 = \lambda(2n+1) \Rightarrow [M]_t = \int_0^t [\lambda(2X_s + 1) - 2X_s\lambda]\, ds = \lambda t$.

2.a)

b) $\begin{pmatrix} 0&1&1&1&2&2 \\ 1&0&1&2&1&2 \\ 1&1&0&2&2&1 \\ 1&2&2&0&3&3 \\ 2&1&2&3&0&3 \\ 2&2&1&3&3&0 \end{pmatrix}$ $\mathrm{diam}(G) = 3$, $L(G) = \frac{54}{30} = \frac{9}{5}$

c) $d = (3,3,3,1,1,1)$
$p(k) = (0, \frac{1}{2}, 0, \frac{1}{2}, 0, \ldots)$ $\langle k \rangle = \frac{12}{6} = 2$

d) $C = \frac{3 \cdot 1}{3 \cdot 3} = 1/3$ $C_1 = C_2 = C_3 = \frac{1}{3}$, $C_4 = C_5 = C_6 = 0 \Rightarrow \langle C \rangle = \frac{1}{6} \cdot 3 \cdot 1/3 = 1/6$

e) $q(1,3) = q(3,1) = 1/4$, $q(3,3) = 1/2$; $q(1) = 1/4$, $q(3) = 3/4$;
$q(3|1) = 1$, $q(3|3) = 2/3$, $q(1|3) = 1/3$, $k_{nn}(1) = 1$, $k_{nn}(3) = 1/3 + 3 \cdot 2/3 = 7/3$

3. b) B_t cont. paths $\Rightarrow X_t$ cont. paths. $\mathbb{E}[X_t] = \frac{1}{\lambda}\mathbb{E}[B_{t\lambda^2}] = 0$ $\forall t \geq 0$
$\mathbb{E}[X_t X_s] = \frac{1}{\lambda^2}\mathbb{E}[B_{t\lambda^2} B_{s\lambda^2}] = \frac{\lambda^2}{\lambda^2}\min\{s, t\}$; $X_t \sim \frac{1}{\lambda} B_{t\lambda^2} \sim N(0, t)$ Gaussian

4. **Birth-death processes**
A general birth-death process $(X_t : t \geq 0)$ is a continuous-time Markov chain with state space $S = \mathbb{N}_0 = \{0, 1, \ldots\}$ and jump rates

$$x \xrightarrow{\alpha_x} x+1 \quad \text{for all } x \in S\,, \qquad x \xrightarrow{\beta_x} x-1 \quad \text{for all } x \geq 1\,.$$

(a) Give the generator G as a matrix and as an operator, and write the master equation in explicit form, i.e. $\frac{d}{dt}\pi_t(x) = \ldots$ ($x = 0$ may need special consideration).
Under which conditions on the jump rates is the process irreducible?

(b) Using detailed balance, find a formula for the stationary probabities $\pi(x)$ in terms of the jump rates and $\pi(0)$, normalization is not required.

(c) Suppose $\alpha_x = \alpha > 0$ for $x \geq 0$ and $\beta_x = x\beta$ for $x \geq 1$ with $\beta > 0$.
Under which conditions on α and β can the stationary probabilities $\pi(x)$ you found in (b) be normalized?
In that case compute the normalization and give a formula for $\pi(x)$.

(d) Suppose $\alpha_x = \beta_x = 2^x$ for $x \geq 1$ and $\alpha_0 = 1$.
Can the stationary probabilities $\pi(x)$ you found in (b) be normalized?
If yes, compute the normalization and give a formula for $\pi(x)$.
Give the transition probabilities of the corresponding jump chain $(Y_n : n \in \mathbb{N}_0)$.
Does it have a stationary distribution? If yes, give a formula.

[20]

5. Consider an even number L of individuals, each having one of two possible types denoted by $X_t(i) \in \{A, B\}$ for all $i = 1, \ldots, L$ and continuous times $t \geq 0$. Each individual changes its type independently of all others at rate 1, in short $\quad A \xrightarrow{1} B \quad$ and $\quad B \xrightarrow{1} A\,.$

(a) Denoting by $X_t = (X_t(i) : i = 1, \ldots, L)$ the vector of types, give the state space of the process $(X_t : t \geq 0)$. Is this process irreducible? Does it have absorbing states?

From now on consider $N_t := \sum_{i=1}^{L} \delta_{X_t(i),A}$ to be the number individuals of type A at time t.

(b) Give state space and generator of the process $(N_t : t \geq 0)$. Is it irreducible?
Show that the stationary distribution is of binomial form and give the parameters.

(c) Consider the rescaled process $U_t^L := \frac{1}{L} N_t$ on the state space $[0, 1]$.
Write down the generator of $(U_t^L : t \geq 0)$ and compute its limit as $L \to \infty$.
Use this to show that the limit process $U_t := \lim_{L\to\infty} U_t^L$ is deterministic and is given as a solution to the ODE $\quad \frac{d}{dt} U_t = 1 - 2U_t\,.$
Solve this ODE for general initial condition $U_0 \in [0, 1]$.

(d) Now take $N_0 = L/2$ and consider the 'fluctuation process' $\quad Z_t^L = \frac{N_t - L/2}{\sqrt{L}} \in \mathbb{R}\,.$
Write down the generator of this process, and use this to show that $(Z_t^L : t \geq 0)$ converges as $L \to \infty$ to an Ornstein-Uhlenbeck process $(Z_t : t \geq 0)$ with generator

$$\mathcal{L}f(z) = -2zf'(z) + \frac{1}{2}f''(z)\,.$$

[20]

a) $\mathcal{L}f(x) = \alpha_x[f(x+1) - f(x)] + \beta_x[f(x-1) - f(x)]\,, x \geq 1 \quad \mathcal{L}f(0) = \alpha_0[f(1) - f(0)]$

$G = \begin{pmatrix} -\alpha_0 & \alpha_0 & \\ \beta_1 & -\alpha_1-\beta_1 & \alpha_1 \\ & & \ddots \end{pmatrix}$ $\quad \frac{d}{dt}\pi_t(x) = \beta_{x+1}\pi_t(x+1) + \alpha_{x-1}\pi_t(x-1) - (\alpha_x + \beta_x)\pi_t(x)\,, x \geq 1$

$\frac{d}{dt}\pi_t(0) = \beta_1 \pi_t(1) - \alpha_0 \pi_t(0)$

irreducible if $\alpha_x > 0\ \forall x \geq 0$; $\beta_x > 0\ \forall x \geq 1$

b) det bal.: $\pi(x)\alpha_x = \pi(x+1)\beta_{x+1} \Rightarrow \pi(x) = \prod_{k=1}^{x} \frac{\alpha_{k-1}}{\beta_k}\pi(0)$

c) $\pi(x) = \left(\frac{\alpha}{\beta}\right)^x \frac{1}{x!}\pi(0) \Rightarrow \pi(0) = \left(\sum_{x\geq 0}\left(\frac{\alpha}{\beta}\right)^x \frac{1}{x!}\right)^{-1} = e^{-\alpha/\beta} \Rightarrow \pi(x) = \left(\frac{\alpha}{\beta}\right)^x \frac{1}{x!} e^{-\alpha/\beta}$ $\forall \alpha, \beta > 0$

d) $\frac{\alpha_{k-1}}{\beta_k} = \frac{1}{2} \Rightarrow \pi(x) = \frac{1}{2^x}\pi(0) \Rightarrow \pi(0) = \left(\sum_{x\geq 0}\frac{1}{2^x}\right)^{-1} = \frac{1}{2} \Rightarrow \pi(x) = \frac{1}{2^{x+1}}$ (yes)

jump chain $p^Y(x, x+1) = p^Y(x, x-1) = \frac{1}{2}$, $x \geq 1$, $p^Y(0,1) = 1$

from det. bal. $\tilde{\pi} = (1/2, 1, 1, \ldots)$ is stat. $\Rightarrow$ cannot be normalized $\Rightarrow$ no stat. distr.

5.a) $S = \{A, B\}^L$. Is irreducible, no abs. states

b) $S = \{0, \ldots, L\}$. $\mathcal{L}f(n) = (L-n)[f(n+1) - f(n)] + n[f(n-1) - f(n)]$
is irreducible $\qquad$ type $B \to A$ $\qquad$ type $A \to B$

by symmetry $Bin(L, 1/2)$, i.e. $\pi(n) = \binom{L}{n}(1/2)^L$

proof by det. bal.: $(n+1)\pi(n+1) = \binom{L}{n+1}(n+1)(1/2)^L = \binom{L}{n}(L-n)(1/2)^L = (L-n)\pi(n)$

c) $\mathcal{L}f\left(\frac{n}{L}\right) = (L-n)\left[f'\left(\frac{n}{L}\right)\frac{1}{L} + O\left(\frac{1}{L^2}\right)\right] + n\left[-f'\left(\frac{n}{L}\right)\frac{1}{L} + O\left(\frac{1}{L^2}\right)\right]$ as $L \to \infty$, $n/L \to u$

$\mathcal{L}f(u) = (1-u)f'(u) - uf'(u) = (1-2u)f'(u)$

Diff. process with $\sigma^2 = 0 \Rightarrow$ deterministic and $dU_t = (1-2U_t)dt$

$\Rightarrow U_t = 1/2 + e^{-2t}(U_0 - 1/2)$

d) $\mathcal{L}f\Big(\underbrace{\frac{n - L/2}{\sqrt{L}}}_{=: z^L}\Big) = (L-n)\left[\frac{1}{\sqrt{L}}f'(z^L) + \frac{1}{2L}f''(z^L) + o\left(\frac{1}{L}\right)\right]$
$+ n\left[-\frac{1}{\sqrt{L}}f'(z^L) + \frac{1}{2L}f''(z^L) + o\left(\frac{1}{L}\right)\right]$

$n = L/2 + \sqrt{L}z^L$, $L - n = L/2 - \sqrt{L}z^L$; as $L, n \to \infty$, $z^L \to z$

$\mathcal{L}f(z) = -2zf'(z) + \frac{1}{2}f''(z)$