MA933 - Networks and Random Processes

MSc in Mathematics of Systems

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These notes and other information about the course are available on

`www2.warwick.ac.uk/fac/sci/mathsys/courses/msc/ma933/`

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References

- G. Grimmett, D. Stirzaker: Probability and Random Processes (3rd edition), Oxford 2001
- C.W. Gardiner: Handbook of Stochastic Methods (3rd edition), Springer 2004
- G. Grimmett: Probability on Graphs, CUP 2010
  `http://www.statslab.cam.ac.uk/~grg/books/pgs.html`
- M.E.J. Newman: Networks: An Introduction, OUP 2010

1. Probability

- **probability space** Ω $\quad\big($ e.g. $\{H,T\},\ \{H,T\}^N,\ \{\text{paths of a stoch. process}\}\big)$
- **events** $A \subseteq \Omega$ $\quad$ (measurable) subsets $\quad$ (e.g. odd numbers on a die)
  $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is the set of all events (subset of the powerset)

Definition 1.1

A **probability distribution** $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0,1]$ which is

(i) normalized, i.e. $\quad \mathbb{P}[\emptyset] = 0 \quad$ and $\quad \mathbb{P}[\Omega] = 1$

(ii) additive, i.e. $\quad \mathbb{P}\big[\cup_i A_i\big] = \sum_i \mathbb{P}[A_i]$,
where $A_1, A_2, \ldots$ is a collection of disjoint events, i.e. $A_i \cap A_j = \emptyset$ for all i, j.

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **probability space**.

- For **discrete** Ω: $\quad \mathcal{F} = \mathcal{P}(\Omega) \quad$ and $\quad \mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega]$
  e.g. $\quad \mathbb{P}[\text{even number on a die}] = \mathbb{P}[2] + \mathbb{P}[4] + \mathbb{P}[6] = 1/2$
- For **continuous** Ω $\quad$ (e.g. $[0,1]$): $\quad \mathcal{F} \subsetneq \mathcal{P}(\Omega)$

1. Independence and conditional probability

- Two events $A, B \subseteq \Omega$ are called **independent** if $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$.

  Example. rolling a die repeatedly
- If $\mathbb{P}[B] > 0$ then the **conditional probability** of A given B is

$$\mathbb{P}[A|B] := \mathbb{P}[A \cap B]/\mathbb{P}[B] \ .$$

  If A and B are independent, then $\mathbb{P}[A|B] = \mathbb{P}[A]$.

Lemma 1.1 (Law of total probability)

Let $B_1, \ldots, B_n$ be a **partition** of Ω such that $\mathbb{P}[B_i] > 0$ for all i. Then

$$\mathbb{P}[A] = \sum_{i=1}^{n} \mathbb{P}[A \cap B_i] = \sum_{i=1}^{n} \mathbb{P}[A|B_i]\,\mathbb{P}[B_i] \ .$$

Note that also $\mathbb{P}[A|C] = \sum_{i=1}^{n} \mathbb{P}[A|C \cap B_i]\,\mathbb{P}[B_i|C]$ provided $\mathbb{P}[C] > 0$.

1. Random variables

Definition 1.2

A **random variable** X is a (measurable) function $X : \Omega \to \mathbb{R}$.

The **distribution function** of the random variable is

$$F(x) = \mathbb{P}[X \leq x] = \mathbb{P}\big[\{\omega : X(\omega) \leq x\}\big] \ .$$

X is called **discrete**, if it only takes values in a countable subset $\{x_1, x_2, \ldots\}$ of $\mathbb{R}$, and its distribution is characterized by the **probability mass function**

$$\pi(x_k) := \mathbb{P}[X = x_k] \ , \quad k = 1, 2, \ldots \ .$$

X is called **continuous**, if its distribution function is

$$F(x) = \int_{-\infty}^{x} f(y)\, dy \quad \text{for all } x \in \mathbb{R} \ ,$$

where $f : \mathbb{R} \to [0, \infty)$ is the **probability density function (PDF)** of X.

1. Random variables

- In general, $f = F'$ is given by the derivative (exists for cont. rv's).
  For discrete rv's, F is a step function with 'PDF'

$$f(x) = F'(x) = \sum_k \pi(x_k)\delta(x - x_k) \ .$$

- The **expected value** of X is given by

$$\mathbb{E}[X] = \begin{cases} \sum_\omega X(\omega)\,\mathbb{P}[\omega] = \sum_k x_k \pi(x_k) \\ \int_\Omega X(\omega) d\mathbb{P}(\omega) = \int_{\mathbb{R}} x f(x)\,dx \end{cases}$$

- The **variance** is given by $\quad \mathrm{var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$
- Two random variables X, Y are independent if the events $\{X \leq x\}$ and $\{Y \leq y\}$ are independent for all $x, y \in \mathbb{R}$. This implies for **joint distributions**

$$f(x,y) = f^X(x) f^Y(y) \quad \text{or} \quad \pi(x_k, x_l) = \pi^X(x_k)\,\pi^Y(x_l)$$

where $\quad f^X(x) = \int_{\mathbb{R}} f(x,y)\,dy \quad$ and $\quad \pi^X(x_k) = \sum_l \pi(x_k, x_l) \quad$ are the **marginals**.

1. Simple random walk

Definition 1.3

Let $X_1, X_2, \ldots \in \{-1, 1\}$ be a sequence of independent, identically distributed random variables **(iidrv's)** with

$$p = \mathbb{P}[X_i = 1] \quad \text{and} \quad q = \mathbb{P}[X_i = -1] = 1 - p \, .$$

The sequence $Y_0, Y_1, \ldots$ defined as $\quad Y_0 = 0 \quad$ and $\quad Y_n = \sum_{k=1}^{n} X_k$

is called the **simple random walk (SRW)** on $\mathbb{Z}$.

- for a single **increment** X_k we have

$$\mathbb{E}[X_k] = p - q = 2p - 1 \, , \quad \operatorname{var}[X_k] = p + q - (p - q)^2 = 4p(1 - p)$$

- $\mathbb{E}[Y_n] = \mathbb{E}\left[\sum_{k=1}^{n} X_k\right] = \sum_{k=1}^{n} \mathbb{E}[X_k] = n(2p - 1)$
  (expectation is a linear operation)
- $\operatorname{var}[Y_n] = \operatorname{var}\left[\sum_{k=1}^{n} X_k\right] = \sum_{k=1}^{n} \operatorname{var}[X_k] = 4np(1 - p)$
  (for a sum of **independent** rv's the variance is additive)

1. LLN and CLT

Theorem 1.2 (Weak law of large numbers (LLN))

Let $X_1, X_2, \ldots \in \mathbb{R}$ be a sequence of iidrv's with $\mu := \mathbb{E}[X_k] < \infty$ and $\mathbb{E}[|X_k|] < \infty$. Then

$$\frac{1}{n} Y_n = \frac{1}{n} \sum_{k=1}^{n} X_k \to \mu \quad \text{as } n \to \infty$$

in distribution (i.e. the distr. fct. of Y_n converges to $\mathbb{1}_{[\mu,\infty)}(x)$ for $x \neq \mu$) .

Theorem 1.3 (Central limit theorem (CLT))

Let $X_1, X_2, \ldots \in \mathbb{R}$ be a sequence of iidrv's with $\mu := \mathbb{E}[X_k] < \infty$ and $\sigma^2 := \mathrm{var}[X_k] < \infty$. Then

$$\frac{Y_n - n\mu}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (X_k - \mu) \to \xi \quad \text{as } n \to \infty$$

in distr., where $\xi \sim N(0, \sigma^2)$ is a **Gaussian** with PDF $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)}$.

Expansion. as $n \to \infty$, $\sum_{k=1}^{n} X_k = n\mu + \sqrt{n}\sigma\xi + o(\sqrt{n})$, $\xi \sim N(0,1)$

1. Discrete-time Markov processes

Definition 1.4

A **discrete-time stochastic process** with **state space** S is a sequence $Y_0, Y_1, \ldots = (Y_n : n \in \mathbb{N}_0)$ of random variables taking values in S.

The process is called **Markov**, if for all $A \subseteq S$, $n \in \mathbb{N}_0$ and $s_0, \ldots, s_n \in S$

$$\mathbb{P}(Y_{n+1} \in A | Y_n = s_n, \ldots, Y_0 = s_0) = \mathbb{P}(Y_{n+1} \in A | Y_n = s_n) \ .$$

A Markov process (MP) is called **homogeneous** if for all $A \subseteq S$, $n \in \mathbb{N}_0$ and $s \in S$

$$\mathbb{P}(Y_{n+1} \in A | Y_n = s) = \mathbb{P}(Y_1 \in A | Y_0 = s) \ .$$

If S is discrete, the MP is called a **Markov chain (MC)**.

The generic probability space Ω is the **path space**

$$\Omega = D(\mathbb{N}_0, S) := S^{\mathbb{N}_0} = S \times S \times \ldots$$

which is uncountable even when S is finite. For a given $\omega \in \Omega$ the function $n \mapsto Y_n(\omega)$ is called a **sample path**.

Up to finite time N and with finite S, $\Omega_N = S^{N+1}$ is finite.

1. Discrete-time Markov processes

Examples.

- For the simple random walk we have state space $S = \mathbb{Z}$ and $Y_0 = 0$. Up to time N, $\mathbb{P}$ is a distribution on the finite path space Ω_N with

$$\mathbb{P}(\omega) = \delta_{0,Y_0(\omega)} \prod_{n=1}^{N} \left(p\, \delta_{1,Y_n(\omega)-Y_{n-1}(\omega)} + q\, \delta_{-1,Y_n(\omega)-Y_{n-1}(\omega)} \right)$$

$$= \begin{cases} p^{\#\text{ of up-steps}}\, q^{\#\text{ of down-steps}} & , \text{ path possible} \\ 0 & , \text{ path not possible} \end{cases}$$

  There are only 2^N paths in Ω_N with non-zero probability.
  For $p = q = 1/2$ they all have the same probability $(1/2)^N$.

- For the generalized random walk with $Y_0 = 0$ and increments $Y_{n+1} - Y_n \in \mathbb{R}$, we have $S = \mathbb{R}$ and $\Omega_N = \mathbb{R}^N$ with an uncountable number of possible paths.
- A sequence $Y_0, Y_1, \ldots \in S$ of iidrv's is also a Markov process with state space S.
- Let $S = \{1, \ldots, 52\}$ be a deck of cards, and $Y_1, \ldots, Y_{52}$ be the cards drawn at random without replacement. Is this a Markov process?

Proposition 1.4

Let $(X_n : n \in \mathbb{N}_0)$ by a homogeneous DTMC with state space S. Then the **transition function**

$$p_n(x,y) := \mathbb{P}[X_n = y | X_0 = x] = \mathbb{P}[X_{k+n} = y | X_k = x] \quad \text{for all } k \geq 0$$

is well defined and fulfills the **Chapman Kolmogorov equations**

$$p_{k+n}(x,y) = \sum_{z \in S} p_k(x,z)\, p_n(z,y) \quad \text{for all } k,n \geq 0,\ x,y \in S\,.$$

Proof. We use the law of total probability, the Markov property and homogeneity

$$\begin{aligned}
\mathbb{P}[X_{k+n} = y | X_0 = x] &= \sum_{z \in S} \mathbb{P}[X_{k+n} = y | X_k = z,\, X_0 = x]\, \mathbb{P}[X_k = z | X_0 = x] \\
&= \sum_{z \in S} \mathbb{P}[X_{k+n} = y | X_k = z]\, \mathbb{P}[X_k = z | X_0 = x] \\
&= \sum_{z \in S} \mathbb{P}[X_n = y | X_0 = z]\, \mathbb{P}[X_k = z | X_0 = x]
\end{aligned}$$

1. Markov chains

- In matrix form with $P_n = \big(p_n(x,y) : x,y \in S\big)$ the Chapman Kolmogorov equations read

$$P_{n+k} = P_n\, P_k \quad \text{and in particular} \quad P_{n+1} = P_n\, P_1 \ .$$

  With $P_0 = \mathbb{I}$, the obvious solution to this recursion is

$$P_n = P^n \quad \text{where we write} \quad P_1 = P = \big(p(x,y) : x,y \in S\big) \ .$$

- The **transition matrix** P and the initial condition $X_0 \in S$ completely determine a homogeneous DTMC, since for all $k \geq 1$ and all events $A_1, \ldots, A_k \subseteq S$

$$\mathbb{P}[X_1 \in A_1, \ldots, X_k \in A_k] = \sum_{s_1 \in A_1} \cdots \sum_{s_k \in A_k} p(X_0, s_1) p(s_1, s_2) \cdots p(s_{k-1}, s_k) \ .$$

- Fixed X_0 can be replaced by an **initial distribution** $\pi_0(x) := \mathbb{P}[X_0 = x]$.

  The distribution at time n is then

$$\pi_n(x) = \sum_{y \in S} \sum_{s_1 \in S} \cdots \sum_{s_{n-1} \in S} \pi_0(y) p(y, s_1) \cdots p(s_{n-1}, x) \quad \text{or} \quad \boldsymbol{\pi_n = \pi_0 P^n} \ .$$

1. Transition matrices

The transition matrix P is **stochastic**, i.e.

$$p(x,y) \in [0,1] \quad \text{and} \quad \sum_y p(x,y) = 1 \,,$$

or equivalently, the column vector $|1\rangle = (1,\ldots,1)^T$
is **eigenvector** with **eigenvalue** 1: $P|1\rangle = |1\rangle$

Example 1 (Random walk with boundaries)

Let $(X_n : n \in \mathbb{N}_0)$ be a SRW on $S = \{1,\ldots,L\}$ with $p(x,y) = p\delta_{y,x+1} + q\delta_{y,x-1}$.
The boundary conditions are

- **periodic** if $p(L,1) = p \,, \quad p(1,L) = q \,,$
- **absorbing** if $p(L,L) = 1 \,, \quad p(1,1) = 1 \,,$
- **closed** if $p(1,1) = q \,, \quad p(L,L) = p \,,$
- **reflecting** if $p(1,2) = 1 \,, \quad p(L,L-1) = 1 \,.$

1. Stationary distributions

Definition 1.5

Let $(X_n : n \in \mathbb{N}_0)$ be a homogeneous DTMC with state space S. The distribution $\pi(x),\ x \in S$ is called **stationary** if for all $y \in S$

$$\sum_{x \in S} \pi(x) p(x,y) = \pi(y) \quad \text{or} \quad \boldsymbol{\pi P = \pi} \ .$$

π is called **reversible** if it fulfills the **detailed balance** conditions

$$\pi(x) p(x,y) = \pi(y) p(y,x) \quad \text{for all } x, y \in S \ .$$

- reversibility implies stationarity, since

$$\sum_{x \in S} \pi(x) p(x,y) = \sum_{x \in S} \pi(y) p(y,x) = \pi(y) \ .$$

- Stationary distributions as row vectors $\langle \pi | = (\pi(x) : x \in S)$ are **left eigenvectors** with **eigenvalue** 1: $\langle \pi | = \langle \pi | P$.

1. Distribution at time n

Consider a DTMC on a finite state space with $|S| = L$, and let $\lambda_1, \ldots, \lambda_L \in \mathbb{C}$ be the **eigenvalues** of the transition matrix P with corresponding

$$\textbf{left (row) eigenvectors } \langle u_i| \quad \text{and} \quad \textbf{right (column) eigenvectors } |v_i\rangle$$

in bra-ket notation. Assuming that **all eigenvalues are distinct** we have

$$P = \sum_{i=1}^{L} \lambda_i |v_i\rangle\langle u_i| \quad \text{and} \quad P^n = \sum_{i=1}^{L} {\lambda_i}^n |v_i\rangle\langle u_i|$$

since eigenvectors can be chosen **orthonormal** $\langle u_i|v_j\rangle = \delta_{i,j}$.

Since $\boldsymbol{\pi}_n = \boldsymbol{\pi}_0 P^n$ we get

$$\langle \boldsymbol{\pi}_n| = \langle \boldsymbol{\pi}_0|v_1\rangle {\lambda_1}^n \langle u_1| + \ldots + \langle \boldsymbol{\pi}_0|v_L\rangle {\lambda_L}^n \langle u_L| \, .$$

- The **Gershgorin theorem** implies that $|\lambda_i| \leq 1$ and contributions with $|\lambda_i| < 1$ decay exponentially (see hand-out 1).
- $\lambda_1 = 1$ corresponds to the stationary distribution and $|v_1\rangle = |1\rangle = (1, \ldots, 1)^T$.
- Other eigenvalues with $|\lambda_i| = 1$ and $\lambda_i \neq 1$ correspond to persistent oscillations.

1. Lazy Markov chains

Definition 1.6

Let $(X_n : n \in \mathbb{N}_0)$ be a DTMC with transition matrix $p(x, y)$. The DTMC with transition matrix

$$p^{\epsilon}(x, y) = \epsilon\delta_{x,y} + (1 - \epsilon)\, p(x, y)\ , \quad \epsilon \in (0, 1)$$

is called a **lazy version** of the original chain.

- Since all diagonal elements are bounded below by $\epsilon > 0$, the Gershgorin theorem now implies for the eigenvalues of P^{ϵ}

$$|\lambda_i| = 1 \quad \Rightarrow \quad \lambda_i = 1\ .$$

  Such a matrix P^{ϵ} is called **aperiodic**, and there are no persistent oscillations.
- The stationary distribution is unique if and only if the eigenvalue $\lambda = 1$ has multiplicity 1, which is independent of lazyness and is discussed later.

1. Absorbing states

Definition 1.7

A state $s \in S$ is called **absorbing** for a DTMC with transition matrix $p(x,y)$, if

$$p(s,y) = \delta_{s,y} \quad \text{for all } y \in S\,.$$

RW with absorbing BC.

Let h_k be the **absorption probability** for $X_0 = k \in S = \{1,\dots,L\}$,

$$h_k = \mathbb{P}[\text{absorption}|X_0 = k] = \mathbb{P}[X_n \in \{1,L\} \text{ for some } n \geq 0 | X_0 = k]\,.$$

Conditioning on the first jump and using Markov, we have the recursion

$$h_k = p h_{k+1} + q h_{k-1} \quad \text{for } k = 2,\dots,L-1\,; \quad h_1 = h_L = 1\,.$$

Ansatz for solution $h_k = \lambda^k\,, \quad \lambda \in \mathbb{C}$:

$$\lambda = p\lambda^2 + q \quad \Rightarrow \quad \lambda_1 = 1\,, \quad \lambda_2 = q/p$$

General solution of 2nd order linear recursion

$$h_k = a\lambda_1^k + b\lambda_2^k = a + b(q/p)^k\,, \quad a,b \in \mathbb{R}\,.$$

Determine coefficients from boundary condition $\Rightarrow$ $h_k \equiv 1$

2. Continuous-time Markov chains

Definition 2.1

A **continuous-time stochastic process** with **state space** S is a family $(X_t : t \geq 0)$ of random variables taking values in S. The process is called **Markov**, if for all $A \subseteq S$, $n \in \mathbb{N}$, $t_1 < \ldots < t_{n+1} \in [0, \infty)$ and $s_1, \ldots, s_n \in S$

$$\mathbb{P}(X_{t_{n+1}} \in A | X_{t_n} = s_n, \ldots, X_{t_1} = s_1) = \mathbb{P}(X_{t_{n+1}} \in A | X_{t_n} = s_n) \, .$$

A Markov process (MP) is called **homogeneous** if for all $A \subseteq S$, $t, u > 0$ and $s \in S$

$$\mathbb{P}(X_{t+u} \in A | X_u = s) = \mathbb{P}(X_t \in A | X_0 = s) \, .$$

If S is discrete, the MP is called a continuous-time **Markov chain (CTMC)**.

The generic probability space Ω of a CTMC is the space of **right-continuous paths**

$$\Omega = D([0, \infty), S) := \left\{ X : [0, \infty) \to S \,\middle|\, X_t = \lim_{u \searrow t} X_u \right\}$$

$\mathbb{P}$ is a probability distribution on Ω, which by **Kolmogorov's extension theorem** is fully specified by its **finite dimensional distributions (FDDs)** of the form

$$\mathbb{P}[X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n] \, , \quad n \in \mathbb{N}, \; t_i \in [0, \infty), \; A_i \subseteq S \, .$$

2. Continuous-time Markov chains

Proposition 2.1

Let $(X_t : t \geq 0)$ by a homogeneous CTMC with state space S. Then for all $t \geq 0$ the **transition function**

$$p_t(x,y) := \mathbb{P}[X_t = y | X_0 = x] = \mathbb{P}[X_{t+u} = y | X_u = x] \quad \text{for all } u \geq 0$$

is well defined and fulfills the **Chapman Kolmogorov equations**

$$p_{t+u}(x,y) = \sum_{z \in S} p_t(x,z)\, p_u(z,y) \quad \text{for all } t,u \geq 0,\ x,y \in S\,.$$

In matrix notation $P_t = \big(p_t(x,y) : x,y \in S\big)$ we get

$$P_{t+u} = P_t\, P_u \quad \text{with} \quad P_0 = \mathbb{I}\,.$$

In particular $\frac{P_{t+\Delta t} - P_t}{\Delta t} = P_t \frac{P_{\Delta t} - \mathbb{I}}{\Delta t} = \frac{P_{\Delta t} - \mathbb{I}}{\Delta t} P_t\,,$

taking $\Delta t \searrow 0$ we get the so-called **forward and backward equations**

$$\frac{d}{dt} P_t = P_t G = G P_t\,, \quad \text{where} \quad G = \left.\frac{dP_t}{dt}\right|_{t=0}$$

is called the **generator** of the process (sometimes also Q-matrix).

2. Continuous-time Markov chains

- The solution is given by the matrix exponential

$$P_t = \exp(tG) = \sum_{k=0}^{\infty} \frac{t^k}{k!} G^k = \mathbb{I} + tG + \frac{t^2}{2} G^2 + \dots \tag{2.1}$$

- The distribution π_t at time time $t > 0$ is then given by

$$\pi_t = \pi_0 \exp(tG) \quad \text{which solves} \quad \frac{d}{dt}\pi_t = \pi_t G \,. \tag{2.2}$$

- On a finite state space with $\lambda_1, \dots, \lambda_L \in \mathbb{C}$ being eigenvalues of G, P_t has eigenvalues $\exp(t\lambda_i)$ with the same eigenvectors $\langle v_i|$, $|u_i\rangle$.

  If the λ_i are distinct, we can expand the initial condition in the eigenvector basis

$$\langle \pi_0| = \alpha_1 \langle v_1| + \dots + \alpha_L \langle v_L|$$

  where $\alpha_i = \langle \pi_0 | u_i \rangle$. This leads to

$$\langle \pi_t| = \alpha_1 \langle v_1| e^{\lambda_1 t} + \dots + \alpha_L \langle v_L| e^{\lambda_L t} \tag{2.3}$$

2. Continuous-time Markov chains

- using (2.1) we have for $G = \big(g(x,y) : x,y \in S\big)$

$$p_{\Delta t}(x,y) = g(x,y)\Delta t + o(\Delta t) \quad \text{for all } x \neq y \in S\,.$$

So $g(x,y) \geq 0$ can be interpreted as transition rates.

$$p_{\Delta t}(x,x) = 1 + g(x,x)\Delta t + o(\Delta t) \quad \text{for all } x \in S\,,$$

and since $\sum_y p_{\Delta t}(x,y) = 1$ this implies that

$$g(x,x) = -\sum_{y \neq x} g(x,y) \leq 0 \quad \text{for all } x \in S\,.$$

- (2.2) can then be written intuitively as the **Master equation**

$$\frac{d}{dt}\pi_t(x) = \underbrace{\sum_{y \neq x} \pi_t(y) g(y,x)}_{\text{gain term}} - \underbrace{\sum_{y \neq x} \pi_t(x) g(x,y)}_{\text{loss term}} \quad \text{for all } x \in S\,.$$

- The **Gershgorin theorem** now implies that either $\lambda_i = 0$ or $\mathrm{Re}(\lambda_i) < 0$ for the eigenvalues of G, so there are no persistent oscillations.

2. Sample paths

Sample paths $t \mapsto X_t(\omega)$ are piecewise constant and right-continuous by convention.

For $X_0 = x$, define the **holding time** $W_x := \inf\{t > 0 : X_t \neq x\}$.

Proposition 2.2

$W_x \sim \text{Exp}(|g(x,x)|)$, i.e. it is **exponentially distributed** with mean $1/|g(x,x)|$, and if $|g(x,x)| > 0$ the chain jumps to $y \neq x$ after time W_x with probability $g(x,y)/|g(x,x)|$.

Proof. W_x has the **memoryless property**, i.e. for all $t, u > 0$

$$\mathbb{P}(W_x > t+u | W_x > t) = \mathbb{P}(W_x > t+u | X_t = x) = \mathbb{P}(W_x > u)$$

where we used the Markov property and homogeneity. Therefore

$$\mathbb{P}(W_x > t+u) = \mathbb{P}(W_x > u)\mathbb{P}(W_x > t) \quad \Rightarrow \quad \mathbb{P}(W_x > t) = e^{\gamma t}$$

where $\gamma = \dfrac{d}{dt}\mathbb{P}(W_x > t)\Big|_{t=0} = \lim\limits_{\Delta t \searrow 0} \dfrac{p_{\Delta t}(x,x) + o(\Delta t) - 1}{\Delta t} = g(x,x) \leq 0$.

Conditioned on leaving the current state shortly, the probability to jump to y is

$$\lim_{\Delta t \searrow 0} \frac{p_{\Delta t}(x,y)}{1 - p_{\Delta t}(x,x)} = \lim_{\Delta t \searrow 0} \frac{\Delta t\, g(x,y)}{1 - 1 - \Delta t\, g(x,x)} = \frac{g(x,y)}{-g(x,x)} .$$

2. Sample paths

- the **jump times** $J_0, J_1, \ldots$ are defined recursively as

$$J_0 = 0 \quad \text{and} \quad J_{n+1} = \inf\{t > J_n : X_t \neq X_{J_n}\} .$$

- due to right-continuous paths, jump times are **stopping times**, i.e. for all $t \geq 0$, the event $\{J_n \leq t\}$ depends only on $(X_s : 0 \leq s \leq t)$.
- By the **strong Markov property** (allows conditioning on state at stopping time), subsequent holding times and jump probabilities are all independent.
- The **jump chain** $(Y_n : n \in N_0)$ with $Y_n := X_{J_n}$ is then a discrete-time Markov chain with transition matrix

$$p^Y(x,y) = \begin{cases} 0 & , \ x = y \\ g(x,y)/|g(x,x)| & , \ x \neq y \end{cases} \text{ if } g(x,x) < 0 \quad \text{and}$$

$$p^Y(x,y) = \delta_{x,y} \text{ if } g(x,x) = 0 \qquad \text{(by convention)} .$$

- A sample path is constructed by simulating the jump chain $(Y_n : n \in \mathbb{N}_0)$ together with independent **holding times** $(W_{Y_n} : n \in \mathbb{N}_0)$, so that $J_n = \sum_{k=0}^{n-1} W_{Y_k}$

2. Examples

- A **Poisson process** with **rate** λ (short PP(λ)) is a CTMC with

$$S = \mathbb{N}_0,\ X_0 = 0 \quad \text{and} \quad g(x,y) = \lambda\delta_{x+1,y} - \lambda\delta_{x,y}\ .$$

  The PP(λ) has **stationary and independent increments** with

$$\mathbb{P}[X_{t+u} = n+k | X_u = n] = p_t(0,k) = \frac{(\lambda t)^k}{k!}\, e^{-\lambda t} \quad \text{for all } u,t > 0,\ k,n \in \mathbb{N}_0$$

  since $\pi_t(k) = p_t(0,k)$ solves the Master equation $\frac{d}{dt}\pi_t(k) = (\pi_t G)(k)$.

- A **birth-death chain** with **birth rates** α_x and **death rates** β_x is a CTMC with

$$S = \mathbb{N}_0 \quad \text{and} \quad g(x,y) = \alpha_x\delta_{x+1,y} + \beta_x\delta_{x-1,y} - (\alpha_x+\beta_x)\delta_{x,y}\ ,$$

  where $\beta_0 = 0$.

  Special cases include

  - $M/M/1$ server queues: $\alpha_x \equiv \alpha > 0$, $\beta_x \equiv \beta > 0$ for $x > 1$
  - $M/M/\infty$ server queues: $\alpha_x \equiv \alpha > 0$, $\beta_x = x\beta$
  - population growth model: $\alpha_x = x\alpha$, $\beta_x = x\beta$

2. Stationary distributions

Definition 2.2

Let $(X_t : t \geq 0)$ be a homogeneous CTMC with state space S. The distribution $\pi(x),\ x \in S$ is called **stationary** if $\langle \boldsymbol{\pi} | G = \langle 0|$, or for all $y \in S$

$$\sum_{x \in S} \pi(x) g(x,y) = \sum_{x \neq y} \big(\pi(x) g(x,y) - \pi(y) g(y,x) \big) = 0 \,. \tag{2.4}$$

π is called **reversible** if it fulfills the **detailed balance conditions**

$$\pi(x) g(x,y) = \pi(y) g(y,x) \quad \text{for all } x, y \in S \,. \tag{2.5}$$

- again, reversibility implies stationarity, since with (2.5) every single summand in (2.4) vanishes
- Stationary distributions are left **eigenvectors** of G with **eigenvalue** 0 .
- $\langle \boldsymbol{\pi} | G = \langle 0| \quad$ implies $\quad \langle \boldsymbol{\pi} | P_t = \langle \boldsymbol{\pi} | \big(\mathbb{I} + \sum_{k \geq 1} t^k G^k / k! \big) = \langle \boldsymbol{\pi} | \quad$ for all $t \geq 0$

2. Stationary distributions

Proposition 2.3 (Existence)

A DTMC or CTMC with finite state space S has at least one stationary distribution.

Proof. Since P and G have row sum 1 and 0 we have $\quad P|\mathbf{1}\rangle = |\mathbf{1}\rangle \quad$ and $\quad G|\mathbf{1}\rangle = |\mathbf{0}\rangle$

So 1 and 0 are eigenvalues, and left eigenvectors can be shown to have non-negative entries and thus can be normalized to be stationary distributions $\langle\boldsymbol{\pi}|$. □

Remark. If S is countably infinite, stationary distributions may not exist, as for example for the SRW on $\mathbb{Z}$ or the PP on $\mathbb{N}$.

Definition 2.3

A CTMC (or DTMC) is called **irreducible**, if for all $x, y \in S$

$$p_t(x,y) > 0 \text{ for some } t > 0 \qquad (p_n(x,y) > 0 \text{ for some } n \in \mathbb{N}) \ .$$

Remark. For continuous time irreducibility implies $p_t(x,y) > 0$ for **all** $t > 0$.

2. Stationary distributions

Proposition 2.4 (Uniqueness)

An irreducible Markov chain has at most one stationary distribution.

Proof. Follows from the **Perron Frobenius theorem:**
Let P be a stochastic matrix ($P = P_t$ for any $t \geq 0$ for CTMCs). Then

1. $\lambda_1 = 1$ is an eigenvalue of P, it is singular if and only if the chain is irreducible. Corresponding left and right eigenvectors have non-negative entries.
2. if the chain is continuous-time or discrete-time aperiodic, all remaining eigenvalues $\lambda_i \in \mathbb{C}$ satisfy $\mathrm{Re}(\lambda_i) < 0$ or $|\lambda_i| < 1$, respectively

The second part of the Perron Frobenius theorem also implies convergence of the transition functions to the stationary distribution, which is usually called ergodicity.

2. Ergodicity

Theorem 2.5 (Ergodicity)

An irreducible (aperiodic) MC with finite state space is **ergodic**, i.e. it has a unique stationary distribution $\boldsymbol{\pi}$ and

$$p_t(x,y) = \mathbb{P}[X_t = y | X_0 = x] \to \pi(y) \quad \text{as } t \to \infty\,, \quad \text{for all } x, y \in S\,.$$

Theorem 2.6 (Ergodic Theorem)

Consider an ergodic Markov chain with unique stationary distribution $\boldsymbol{\pi}$. Then for every bounded function $f : S \to \mathbb{R}$ we have with probability 1

$$\frac{1}{T}\int_0^T f(X_t)\,dt \quad \text{or} \quad \frac{1}{N}\sum_{n=1}^{N} f(X_n) \;\to\; \mathbb{E}_{\boldsymbol{\pi}}[f] \quad \text{as } T, N \to \infty\,.$$

- for a proof see e.g. [GS], chapter 9.5
- Stationary expectations can be approximated by time averages, which is the basis for a technique called **Markov chain Monte Carlo**.
- for example, choosing the indicator function $f = \mathbb{1}_x$ we get $\mathbb{E}_{\boldsymbol{\pi}}[f] = \pi(x)$

2. Reversibility

Proposition 2.7 (Time reversal)

Let $(X_t : t \in [0,T])$ be a finite state, irreducible CTMC with generator G^X on a compact time interval which is **stationary**, i.e. $X_t \sim \boldsymbol{\pi}$ for $t \in [0,T]$. Then the **time reversed chain**

$$(Y_t : t \in [0,T]) \quad \text{with} \quad Y_t := X_{T-t}$$

is a stationary CTMC with generator $g^Y(x,y) = \dfrac{\pi(y)}{\pi(x)}\, g^X(y,x)$.

- An analogous statement holds for stationary, finite state, irreducible DTMCs with $p^Y(x,y) = \dfrac{\pi(y)}{\pi(x)}\, p^X(y,x)$.
- Stationary chains with reversible $\boldsymbol{\pi}$ are **time-reversible**, $g^Y(x,y) = g^X(x,y)$.
- The definition of stationary chains can be extended to negative times, $(X_t : t \in \mathbb{R})$, with the time reversed chain given by $Y_t := X_{-t}$.
- The time reversal of non-stationary chains is in general **not** a homogeneous MP.

Example. SRW on finite state space

2. Countably infinite state space

For infinite state space, Markov chains can get 'lost at infinity' and have no stationary distribution. Let $T_x := \inf\{t > J_1 : X_t = x\}$ be the first **return time** to a state x.

(For DTMCs return times are defined as $T_x := \inf\{n \geq 1 : X_n = x\}$)

Definition 2.4

A state $x \in S$ is called

- **transient**, if $\mathbb{P}[T_x = \infty | X_0 = x] > 0$
- **null recurrent**, if $\mathbb{P}[T_x < \infty | X_0 = x] = 1$ and $\mathbb{E}[T_x | X_0 = x] = \infty$
- **positive recurrent**, if $\mathbb{P}[T_x < \infty | X_0 = x] = 1$ and $\mathbb{E}[T_x | X_0 = x] < \infty$

- For an irreducible MC all states are either transient, null or positive recurrent. The MC has a unique stationary distribution if and only if it is positive recurrent.
- A transient CTMC can exhibit **explosion**. Define the **explosion time**

$$J_\infty := \lim_{n \to \infty} J_n \in (0, \infty] \quad \text{where } J_n \text{ are the jump times of the chain .}$$

The chain is called **non-explosive** if $\mathbb{P}[J_\infty = \infty] = 1$.
This is always the case if S is finite or $\sup_{x \in S} |g(x,x)| < \infty$.

3. Markov processes with $S = \mathbb{R}$

Proposition 3.1

Let $(X_t : t \geq 0)$ by a homogeneous MP as in Definition 18 with state space $S = \mathbb{R}$. Then for all $t \geq 0$ the **transition kernel** for all $x, y \in \mathbb{R}$

$$P_t(x, dy) := \mathbb{P}[X_t \in dy | X_0 = x] = \mathbb{P}[X_{t+u} \in dy | X_u = x] \quad \text{for all } u \geq 0$$

is well defined. If it is absolutely continuous the **transition density** p_t with

$$P_t(x, dy) = p_t(x, y)\, dy$$

exists and fulfills the **Chapman Kolmogorov equations**

$$p_{t+u}(x, y) = \int_{\mathbb{R}} p_t(x, z)\, p_u(z, y)\, dz \quad \text{for all } t, u \geq 0,\ x, y \in \mathbb{R}\ .$$

As for CTMCs, the transition densities and the initial distribution $p_0(x)$ describe all **finite dimensional distributions (fdds)**

$$\mathbb{P}[X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n] = \int_R dz_0 p_0(z_0) \int_{-\infty}^{x_1} dz_1 p_{t_1}(z_0, z_1) \cdots \int_{-\infty}^{x_n} dz_n p_{t_n - t_{n-1}}(z_{n-1}, z_n)$$

for all $n \in \mathbb{N}$, $0 < t_1 < \ldots < t_n$ and $x_1, \ldots x_n \in \mathbb{R}$.

3. Jump processes

$(X_t : t \geq 0)$ is a **jump process** with state space $S = \mathbb{R}$ characterized by a

jump rate density $r(x,y) \geq 0$ with a uniformly bounded

total exit rate $R(x) = \int_{\mathbb{R}} r(x,y)\,dy < \bar{R} < \infty$ for all $x \in \mathbb{R}$.

Ansatz for transition function as $\Delta t \to 0$:

$$p_{\Delta t}(z,y) = r(z,y)\Delta t + \big(1 - R(z)\Delta t\big)\delta(y-z)$$

Then use the Chapman Kolmogorov equations

$$p_{t+\Delta t}(x,y) - p_t(x,y) = \int_{\mathbb{R}} p_t(x,z)\, p_{\Delta t}(z,y)\, dz - p_t(x,y) =$$

$$= \int_{\mathbb{R}} p_t(x,z) r(z,y)\Delta t\, dz + \int_{\mathbb{R}} \big(1 - 1 - R(z)\Delta t\big) p_t(x,z)\delta(y-z) dz$$

to get the **Kolmogorov-Feller equation**

$$\partial_t p_t(x,y) = \int_{\mathbb{R}} \Big(p_t(x,z) r(z,y) - p_t(x,y) r(y,z) \Big)\, dz\ .$$

As for CTMC sample paths $t \mapsto X_t(\omega)$ are piecewise constant and right-continuous.

3. Gaussian processes

$\mathbf{X} = (X_1, \ldots, X_n) \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ is a **multivariate Gaussian** in $\mathbb{R}^n$ if it has PDF

$$f(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\Big(-\frac{1}{2} \langle \mathbf{x} - \boldsymbol{\mu} | \, \Sigma^{-1} \, | \mathbf{x} - \boldsymbol{\mu} \rangle \Big) ,$$

with **mean** $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ and **covariance matrix**

$$\Sigma = (\sigma_{ij} : i,j = 1, \ldots, n) , \quad \sigma_{ij} = \mathrm{Cov}[X_i, X_j] = \mathbb{E}\big[(X_i - \mu_i)(X_j - \mu_j)\big] .$$

Definition 3.1

A stochastic process $(X_t : t \geq 0)$ with state space $S = \mathbb{R}$ is a **Gaussian process** if for all $n \in \mathbb{N}$, and all $t_1, \ldots, t_n \geq 0$ the vector $(X_{t_1}, \ldots, X_{t_n})$ has a multivariate Gaussian distribution.

Proposition 3.2

All fdds of a Gaussian process $(X_t : t \geq 0)$ are fully characterized by the **mean** and the **covariance function**

$$m(t) := \mathbb{E}[X_t] \quad \text{and} \quad \sigma(s,t) := \mathrm{Cov}[X_s, X_t] .$$

3. Stationary independent increments

Definition 3.2

A stochastic process $(X_t : t \geq 0)$ has **stationary increments** if

$$X_t - X_s \sim X_{t-s} - X_0 \quad \text{for all } 0 \leq s \leq t \,.$$

It has **independent increments** if for all $n \geq 1$ and $0 \leq t_1 < \cdots < t_n$

$$\{X_{t_{k+1}} - X_{t_k} : 1 \leq k < n\} \quad \text{are independent} \,.$$

Example. The Poisson process $(N_t : t \geq 0) \sim PP(\lambda)$ has stationary independent increments with $N_t - N_s \sim \text{Poi}\big(\lambda(t-s)\big)$.

Proposition 3.3

The following two statements are equivalent for a stochastic process $(X_t : t \geq 0)$:

- X_t has stationary independent increments and $X_t \sim \mathcal{N}(0,t)$ for all $t \geq 0$.
- X_t is a Gaussian process with $m(t) = 0$ and $\sigma(s,t) = \min\{s,t\}$.

Stationary independent increments have **stable distributions** such as Gaussian or Poisson.

3. Brownian motion

Definition 3.3

Standard Brownian motion $(B_t : t \geq 0)$ is a stochastic process that satisfies either of the two equivalent properties in Proposition 3.3 and has **continuous paths**, i.e.

$$\mathbb{P}\Big[\{\omega : t \mapsto B_t(\omega) \text{ is continuous in } t \geq 0\}\Big] = 1 \ .$$

Theorem 3.4 (Wiener)

There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which standard Brownian motion exists.

Proof idea. Construction on $\Omega = \mathbb{R}^{[0,\infty)}$, using **Kolmogorov's extension theorem**: For every 'consistent' description of finite dimensional distributions (fdds) there exists a corresponding stochastic process on Ω, i.e. a 'unique' law $\mathbb{P}$.
The main problem is to show that there exists a 'version' of the process that has continuous paths, i.e. $\mathbb{P}$ can be chosen to concentrate on continuous paths ω.

Remark. Construction of $(N_t : t \geq 0) \sim PP(\lambda)$ is

$$N_t := \max\left\{k \geq 1 : \tau_1 + \cdots + \tau_k \leq t\right\} , \quad \tau_1, \tau_2, \cdots \sim \text{Exp}(\lambda) \text{ iidrvs}$$

3. Properties of Brownian motion

- SBM is a time-homogeneous MP with $B_0 = 0$.
- $\sigma B_t + x$ with $\sigma > 0$ is a (general) BM with $B_t \sim \mathcal{N}(x, \sigma^2 t)$.
  The transition density is given by a Gaussion PDF

$$p_t(x,y) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(y-x)^2}{2\sigma^2 t}\right)$$

  This is also called the **heat kernel**, since it solves the **heat/diffusion equation**

$$\frac{\partial}{\partial t} p_t(x,y) = \frac{\sigma^2}{2} \frac{\partial^2}{\partial y^2} p_t(x,y) \quad \text{with} \quad p_0(x,y) = \delta(y-x)\ .$$

- SBM is **self-similar** with **Hurst exponent** $H = 1/2$, i.e.

$$(B_{\lambda t} : t \geq 0) \sim \lambda^H (B_t : t \geq 0) \quad \text{for all } \lambda > 0\ .$$

- $t \mapsto B_t$ is $\mathbb{P} - a.s.$ **not differentiable** at t for all $t \geq 0$.
  For fixed $h > 0$ define $\xi_t^h := (B_{t+h} - B_t)/h \sim \mathcal{N}(0, 1/h)$, which is a mean-0 Gaussian process with covariance $\sigma(s,t) = \begin{cases} 0 & , \ |t-s| > h \\ (h - |t-s|)/h^2 & , \ |t-s| < h \end{cases}$.
  The (non-existent) derivative $\xi_t := \lim_{h \to 0} \xi_t^h$ is called **white noise** and is formally a mean-0 Gaussian process with covariance $\sigma(s,t) = \delta(t-s)$.

3. Generators as operators

For a CTMC $(X_t : t \geq 0)$ with state space S we have for $f : S \to \mathbb{R}$

$$\frac{d}{dt}\mathbb{E}\big[f(X_t)\big] = \frac{d}{dt}\langle \pi_t | f \rangle = \frac{d}{dt}\langle \pi_0 | P_t | f \rangle = \langle \pi_0 | P_t G | f \rangle = \langle \pi_t | G | f \rangle = \mathbb{E}\big[(Gf)(X_t)\big] \ .$$

The **generator** G can be defined as an operator $G : C(S) \to C(S)$

$$G|f\rangle(x) = (Gf)(x) = \sum_{\substack{y \in S \\ y \neq x}} g(x,y)\big[f(y) - f(x)\big] \ .$$

For **Brownian motion** we have $\dfrac{d}{dt}\langle \pi_t | f \rangle = \langle \pi_t | \mathcal{L} | f \rangle$, where for $f \in C^2(\mathbb{R})$

the **generator of BM** is $(\mathcal{L}f)(x) = \dfrac{\sigma^2}{2}\,\Delta f(x) \qquad \left(\text{or } \dfrac{\sigma^2}{2} f''(x)\right) .$

For **jump processes** with $S = \mathbb{R}$ and rate density $r(x,y)$ the generator is

$$(\mathcal{L}f)(x) = \int_{\mathbb{R}} r(x,y)\big[f(y) - f(x)\big]\, dy \ .$$

3. Brownian motion as scaling limit

Proposition 3.5

Let $(X_t : t \geq 0)$ be a jump process with translation invariant rates $r(x, y) = q(y - x)$ which have

$$\text{mean zero} \qquad \int_{\mathbb{R}} q(z)\, z\, dz = 0 \quad \text{and}$$

$$\text{finite second moment} \qquad \sigma^2 := \int_{\mathbb{R}} q(z)\, z^2\, dz < \infty\ .$$

Then for all $T > 0$ the rescaled process

$$\left(\epsilon X_{t/\epsilon^2} : t \in [0, T]\right) \quad \Rightarrow \quad \left(B_t : t \in [0, T]\right) \quad \text{as } \epsilon \to 0$$

converges in distribution to a BM with generator $\mathcal{L} = \frac{1}{2}\sigma^2 \Delta$ for all $T > 0$.

Proof. Taylor expansion of the generator for test functions $f \in C^3(\mathbb{R})$, and tightness argument for continuity of paths (requires fixed interval $[0, T]$).

3. Diffusion processes

Definition 3.4

A **diffusion process** with **drift** $a(x,t) \in \mathbb{R}$ and **diffusion** $\sigma^2(x,t) > 0$ is a real-valued process with continuous paths and generator

$$(\mathcal{L}f)(x) = a(x,t)f'(x) + \frac{1}{2}\sigma^2(x,t)f''(x) \, .$$

Examples.

- The **Ornstein-Uhlenbeck process** is a diffusion process with generator

$$(\mathcal{L}f)(x) = -\alpha\, x f'(x) + \frac{1}{2}\sigma^2 f''(x) \, , \quad \alpha, \sigma^2 > 0 \, .$$

  It has a Gaussian stationary distribution $\mathcal{N}(0, \sigma^2/(2\alpha))$.
  If the initial distribution π_0 is Gaussian, this is a Gaussian process.

- The **Brownian bridge** is a Gaussian diffusion with $X_0 = 0$ and generator

$$(\mathcal{L}f)(x) = -\frac{x}{1-t}f'(x) + \frac{1}{2}f''(x) \, .$$

  Equivalently, it can be characterized as a SBM conditioned on $B_1 = 0$.

3. Diffusion processes

Time evolution of the mean. Use $\frac{d}{dt}\mathbb{E}[f(X_t)] = \mathbb{E}[(\mathcal{L}f)(x_t)]$ with $f(x) = x$

$$\frac{d}{dt}\mathbb{E}[X_t] = \mathbb{E}[a(X_t, t)]$$

Time evolution of the transition density. With $X_0 = x$ we have for $p_t(x, y)$

$$\int_{\mathbb{R}} \frac{d}{dt} p_t(x, y) f(y) dy = \frac{d}{dt}\mathbb{E}[f(X_t)] = \int_{\mathbb{R}} p_t(x, y)\mathcal{L}f(y) dy \quad \text{for any } f \ .$$

Use integration by parts to get the **Fokker-Planck equation**

$$\frac{d}{dt} p_t(x, y) = -\frac{\partial}{\partial y}\big(a(y, t)p_t(x, y)\big) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\sigma^2(y, t)p_t(x, y)\big) \ .$$

Stationary distributions for time-independent $a(y) \in \mathbb{R}$ and $\sigma^2(y) > 0$

$$\frac{\partial}{\partial y}\big(a(y)p^*(y)\big) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\sigma^2(y)p^*(y)\big) \ ,$$

leads to a stationary density (modulo normalization fixing $p^*(0)$)

$$p^*(x) = p^*(0) \exp\Big(\int_0^x \frac{2a(y) - (\sigma^2)'(y)}{\sigma^2(y)} dy\Big) \ .$$

3. Fluctuations and martingales

Definition 3.5

A real-valued stochastic process $(M_t : t \geq 0)$ is a **martingale** w.r.t. the process $(X_t : t \geq 0)$ if for all $t \geq 0$ we have $\mathbb{E}[|M_t|] < \infty$ and

$$\mathbb{E}\big[M_t \big| \{X_u : 0 \leq u \leq s\}\big] = M_s \quad \text{a.s. for all } s \leq t \ .$$

If in addition $\mathbb{E}[M_t^2] < \infty$, there exists a unique increasing process $([M]_t : t \geq 0)$ called the **quadratic variation**, with $[M]_0 = 0$ and such that $M_t^2 - [M]_t$ is martingale.

Theorem 3.6 (Itô's formula)

Let $(X_t : t \geq 0)$ be a Markov process on state space S with generator $\mathcal{L}$. Then for any smooth enough $f : S \times [0, \infty) \to \mathbb{R}$

$$f(X_t, t) - f(X_0, 0) = \int_0^t (\mathcal{L}f)(X_s, s)ds + \int_0^t \partial_s f(X_s, s)ds + M_t^f \ ,$$

where $(M_t^f : t \geq 0)$ is a martingale w.r.t. $(X_t : t \geq 0)$ with $M_0^f = 0$ and

$$\text{quadratic variation} \quad [M^f]_t = \int_0^t \big((\mathcal{L}f^2)(X_s, s) - 2(f\mathcal{L}f)(X_s, s)\big)ds \ .$$

3. Fluctuations and martingales

- For a Poisson process $(N_t : t \geq 0)$ with rate $\lambda > 0$ Itô's formula implies that

$$M_t := N_t - \lambda t \quad \text{is a martingale with quadr. variation} \quad [M]_t = \lambda t \ .$$

- **Lévy's characterication of BM**: A continuous martingale $(M_t : t \geq 0)$ on $\mathbb{R}$ with $M_0 = 0$ and quadratic variation $[M]_t = t$ is standard Brownian motion.
- Furthermore, any continuous martingale $(M_t : t \geq 0)$ on $\mathbb{R}$ with $M_0 = 0$ is a continuous time-change of a standard Brownian motion, i.e.

$$M_t = B_{[M]_t} \quad \text{for 'some' SBM} \quad (B_t : t \geq 0) \ .$$

- For a diffusion process, choosing $f(X_t, t) = X_t$ in Itô's formula leads to

$$X_t - X_0 = \int_0^t a(X_s, s)ds + M_t \quad \text{with} \quad [M]_t = \int_0^t \sigma^2(X_s, s)ds \ .$$

  Related time-changed BMs can be written as **stochastic Itô integrals**

$$M_t = \int_0^t \sigma(X_s, s)dB_s := B_{[M]_t} \ .$$

  Therefore $\sigma \equiv 0$ implies deterministic dynamics with $M_t \equiv 0$.
- **Stochastic differential equation (SDE)** $\quad dX_t = a(X_t, t)dt + \sigma(X_t, t)dB_t \ .$

3. Beyond diffusion

Definition 3.6

A **Lévy process** $(X_t : t \geq 0)$ is a real-valued process with right-continuous paths and stationary, independent increments.

The generator has a part with constant drift $a \in \mathbb{R}$ and diffusion $\sigma^2 \geq 0$

$$\mathcal{L}f(x) = af'(x) + \frac{\sigma^2}{2} f''(x) + \int_{\mathbb{R}} \Big(f(x+z) - f(x) - zf'(x)\mathbb{1}_{(0,1)}(|z|) \Big) \nu(dz) \ ,$$

and a jump part with positive measure ν which may have a density $\nu(dz) = r(z)dz$, and fulfills $\int_{|z|>1} \nu(dz) < \infty$ and $\int_{0<|z|<1} z^2 \nu(dz) < \infty$.

Examples.

- Diffusion processes, in particular **BM** with $a = 0$, $\sigma^2 > 0$ and $\nu \equiv 0$, or jump processes, in particular **Poisson** with $a = \sigma = 0$ and $\nu(dz) = \lambda\delta(z-1)dz$.
- For $a = \sigma = 0$ and heavy-tailed jump distribution

$$\nu(dz) = \frac{Cdz}{|z|^{1+\alpha}} \quad \text{with} \quad C > 0 \text{ and } \alpha \in (0, 2]$$

the process is called **α-stable symmetric Lévy process** or **Lévy flight**; **super-diffusive behaviour** due to long jumps with infinite mean or variance

3. Beyond diffusion

In general, a process $(X_t : t \geq 0)$ is said to exhibit **anomalous diffusion** if

$$\mathrm{Var}[X_t]/t \to \begin{cases} 0 & , \text{ sub-diffusive} \\ \infty & , \text{ super-diffusive} \end{cases} \qquad \text{as } t \to \infty \, .$$

Definition 3.7

A **fractional Brownian motion (fBM)** $(B_t^H : t \geq 0)$ with **Hurst index** $H \in (0,1)$ is a mean-zero Gaussian process with continuous paths, $B_0^H = 0$ and covariances

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \quad \text{for all } s,t \geq 0 \, .$$

- For $H = 1/2$, fBM is standard Brownian motion.
- fBM has stationary Gaussian increments where for all $t > s \geq 0$

  $$B_t^H - B_s^H \sim B_{t-s}^H \sim \mathcal{N}\left(0, (t-s)^{2H}\right) \, ,$$

  which for $H \neq 1/2$ are **not** independent and the process is **non-Markov**.
- fBM exhibits anomalous diffusion with $\mathrm{Var}[B_t^H] = t^{2H}$.
- fBM is **self-similar**, i.e. $(B_{\lambda t}^H : t \geq 0) \sim \lambda^H (B_t^H : t \geq 0)$ for all $\lambda > 0$.

4. Graphs - definition

Definition 4.1

A **graph** (or **network**) $G = (V, E)$ consists of a finite set $V = \{1, \ldots, N\}$ of **vertices** (or **nodes**, **points**), and a set $E \subseteq V \times V$ of **edges** (or **links**, **lines**).
The graph is called **undirected** if $(i, j) \in E$ implies $(j, i) \in E$, otherwise **directed**.
The structure of the graph is encoded in the **adjacency** (or **connectivity**) **matrix**

$$A = (a_{ij} : i, j \in V) \quad \text{where} \quad a_{ij} = \begin{cases} 1\,, & (i,j) \in E \\ 0\,, & (i,j) \notin E \end{cases}.$$

We denote the number of edges by $K = |E|$ for directed, or $K = |E|/2$ for undirected graphs.

- Graphs we consider do not have self edges, i.e. $(i, i) \notin E$ for all $i \in V$, or multiple edges, since edges (i, j) are unique elements of E.
- **Weighted graphs** with edge weights $w_{ij} \in \mathbb{R}$ can be used to represent continuous- or discrete-time Markov chains.
- In general graphs can also be infinite, but we will focus on finite graphs. Many of the following graph characteristics only make sense in the finite case.

4. Graphs - paths and connectivity

Definition 4.2

A **path** γ_{ij} of length $l = |\gamma_{ij}|$ from vertex i to j is sequence of vertices

$$\gamma_{ij} = (v_1 = i, v_2, \ldots, v_{l+1} = j) \quad \text{with} \quad (v_k, v_{k+1}) \in E \text{ for all } k = 1, \ldots, l \,,$$

and $v_k \neq v_{k'}$ for all $k \neq k' \in \{1, \ldots, l\}$ (i.e. each vertex is visited only once).
If such a path exists, we say that vertex i is **connected** to j (write $i \to j$).
Shortest paths between vertices i, j are called **geodesics** (not necessarily unique) and their length d_{ij} is called the **distance** from i to j. If $i \not\to j$ we set $d_{ij} = \infty$.
A graph is **connected** if $d_{ij} < \infty$ for all $i, j \in V$.
The **diameter** and the **characteristic path length** of the graph G are given by

$$\operatorname{diam}(G) := \max\{d_{ij} : i, j \in V\} \in \mathbb{N}_0 \cup \{\infty\} \,,$$
$$L = L(G) := \frac{1}{N(N-1)} \sum_{i,j \in V} d_{ij} \in [0, \infty] \,.$$

For undirected graphs we have $d_{ij} = d_{ji}$ which is finite if $i \leftrightarrow j$, and they can be decomposed into **connected components**, where we write

$$C_i = \{j \in V : j \leftrightarrow i\} \quad \text{for the component containing vertex } i \,.$$

4. Graphs - degrees

Definition 4.3

The **in-** and **out-degree** of a node $i \in V$ is defined as

$$k_i^{\text{in}} = \sum_{j \in V} a_{ji} \quad \text{and} \quad k_i^{\text{out}} = \sum_{j \in V} a_{ij} \ .$$

$k_1^{\text{in}}, \ldots k_N^{\text{in}}$ is called the **in-degree sequence** and the **in-degree distribution** is

$$\left(p^{\text{in}}(k) : k \in \{0, \ldots, K\}\right) \quad \text{with} \quad p^{\text{in}}(k) = \tfrac{1}{N} \sum_{i \in V} \delta_{k, k_i^{\text{in}}}$$

giving the fraction of vertices with in-degree k. The same holds for out-degrees, and in undirected networks we simply write $k_i = k_i^{\text{in}} = k_i^{\text{out}}$ and $p(k)$.

- Note that $\sum_{i \in V} k_i = \sum_{i,j \in V} a_{ij} = |E|$ (also for directed), average and variance are

$$\langle k \rangle = \tfrac{1}{N} \sum_{i \in V} k_i = |E|/N = \sum_k k p(k) \ , \quad \sigma^2 = \langle k^2 \rangle - \langle k \rangle^2 \ .$$

- In a **regular graph** (usually undirected) all vertices have equal degree $k_i \equiv k$.
- Graphs where the degree distribution shows a power law decay, i.e. $p(k) \propto k^{-\alpha}$ for large k, are often called **scale-free**.
  Real-world networks are often scale-free with exponent around $\alpha \approx 3$.

4. Graphs - first examples

Example 2 (Some graphs)

The **complete graph** K_N with N vertices is an undirected graph where all $N(N-1)/2$ vertices $E = ((i,j) : i \neq j \in V)$ are present.
Regular lattices $\mathbb{Z}^d$ with edges between nearest neighbours are examples of regular graphs with degree $k = 2d$.

Definition 4.4

A **tree** is an undirected graph where any two vertices are connected by exactly one path. Vertices with degree 1 are called **leaves**.
In a **rooted tree** one vertex $i \in V$ is the designated **root**, and the graph can be directed, where all vertices point towards or away from the root.

A **cycle** is a closed path γ_{ii} of length $|\gamma_{ii}| > 2$. G is a tree if and only if

- it is connected and has no cycles;
- it is connected but is not connected if a single edge is removed;
- it has no cycles but a cycle is formed if any edge is added.

4. Graphs - degree correlations

Definition 4.5

For undirected graphs, the **joint degree distribution** of nodes linked by an edge is

$$q(k,k') = \frac{1}{|E|}\sum_{(i,j)\in E}\delta_{k_i,k}\delta_{k_j,k'} = \frac{\sum_{i,j\in V} a_{ij}\delta_{k_i,k}\delta_{k_j,k'}}{\sum_{i,j\in V} a_{ij}} = q(k',k)\ .$$

With the marginal $q(k') = \sum_k q(k,k')$ we have the **conditional degree distribution**

$$q(k|k') = q(k,k')/q(k') \quad \text{with average} \quad k_{nn}(k') := \sum_k kq(k|k')\ .$$

The network is called **uncorrelated** if $k_{nn}(k')$ is independent of k', **assortative** if $k_{nn}(k') \nearrow$ in k' and **disassortative** if $k_{nn}(k') \searrow$ in k'.

- The marginal $q(k)$ corresponds to **edge biased degree sampling**, i.e.

$$q(k) = \sum_{k'} q(k,k') = \frac{1}{|E|}\sum_{i,j\in V} a_{ij}\delta_{k_i,k} = \frac{N}{|E|}\frac{1}{N}\sum_{i\in V} k_i\delta_{k_i,k} = \frac{kp(k)}{\langle k\rangle}\ .$$

  For uncorrelated networks $q(k|k') = q(k)$ and thus $\quad k_{nn}(k') = \langle k^2\rangle/\langle k\rangle$.

4. Subgraphs

- The degree of correlation can be quantified by the **correlation coefficient**

$$\chi := \frac{\langle kk' \rangle_q - \langle k \rangle_q^2}{\langle k^2 \rangle_q - \langle k \rangle_q^2} = \frac{\sum_{k,k'} kk' \left(q(k,k') - q(k)q(k') \right)}{\sum_k k^2 q(k) - \left(\sum_k kq(k) \right)^2} \in [-1, 1] \, .$$

Definition 4.6

A **subgraph** $G' = (V', E')$ of $G = (V, E)$ is a graph such that $V' \subseteq V$ and $E' \subseteq E$.

- Small connected subgraphs are also called **motifs**, the simplest non-trivial examples in undirected graphs are connected triples and triangles.
- Fully connected (complete) subgraphs which are maximal with respect to connectedness are called **cliques**.
- A **spanning tree** is a tree subgraph that contains all vertices of the graph.
- A subgraph G' is called a **community**, if (for example)

$$\sum_{i,j \in V'} a_{ij} > \sum_{i \in V', j \notin V'} a_{ij} \quad \text{(there are also other definitions)} \, .$$

4. Clustering

Clustering aims to quantify the probability that two neighbours of a given vertex are themselves neighbours. Two different definitions are used in the literature.

Definition 4.7

The **global clustering coefficient** for an undirected graph is defined as

$$C = \frac{3 \times \#\text{ of (connected) triangles}}{\#\text{ of (connected) triples}} = \frac{3\sum_{i<j<l} a_{ij}a_{jl}a_{li}}{\sum_{i<j<l}(a_{ij}a_{il} + a_{ji}a_{jl} + a_{li}a_{lj})} \in [0,1] \,.$$

Alternatively, one can define a **local clustering coefficient**

$$C_i = \frac{\#\text{ of triangles containing vertex } i}{\#\text{ of triples centered on vertex } i} = \frac{\sum_{j<l} a_{ij}a_{jl}a_{li}}{\sum_{j<l} a_{ij}a_{il}} \in [0,1] \,,$$

and use the average $\langle C_i \rangle = \frac{1}{N}\sum_i C_i$ to quantify clustering.

- For a tree we have $C = \langle C_i \rangle = 0$ and for the complete graph $C = \langle C_i \rangle = 1$.
- Higher-order clustering coefficients can be defined similarly, using different subgraphs as basis.

4. Graph spectra

Definition 4.8

The **spectral density** of a graph $G = (V, E)$ is

$$\rho(\lambda) := \tfrac{1}{N} \sum_{i \in V} \delta(\lambda - \lambda_i) \quad \text{where} \quad \lambda_1, \ldots, \lambda_N \in \mathbb{C}$$

are the eigenvalues of the adjacency matrix A.

- **Perron-Frobenius:** A has a real eigenvalue $\lambda_1 > 0$ with maximal modulus and real, non-negative eigenvector(s). If the graph is connected, it has multiplicity 1 and $|\lambda_j| < \lambda_1$ for all other eigenvalues with $j \neq 1$.
- For undirected graphs, $(A^n)_{ij}$ is equal to the **number of walks** (paths which allow repeated vertices) from i to j of length n. We also have

$$\mathrm{Tr}(A^n) = \sum_{i=1}^{N} \lambda_i^n \quad \text{and} \quad \big(\mathrm{Tr}(A)\big)^n = 0 \,,$$

which can be used to derive statements like:

$$\sum_{i<j} \lambda_i \lambda_j = -|E| \,, \qquad \sum_{i<j<l} \lambda_i \lambda_j \lambda_l = 2 \cdot \#\text{ of triangles in } G \,.$$

4. Graph Laplacian

Definition 4.9

The **Graph Laplacian** for a graph (V, E) with adjacency matrix A is defined as

$$Q := A - D \quad \text{where} \quad D = \left(\delta_{ij} \sum_{l \neq i} a_{il} : i, j \in V\right).$$

- Q has eigenvalues in $\mathbb{C}$ with real part $\mathrm{Re}(\lambda) < 0$ except for $\lambda_1 = 0$, which follows directly from the Gershgorin theorem and vanishing row sums.
  The **multiplicity of λ_1 equals the number of connected components** in undirected graphs. Properly chosen orthogonal eigenvectors to λ_1 have non-zero entries on the individual connected components.
  The smaller the second largest real part of an eigenvalue, the harder it is to cut G into separated components by removing edges.
- Q defines a generator matrix of a continuous-time random walk on V with transition rates a_{ij}. Using weighted graphs, any finite state CTMC can be represented in this way.
- The Laplacian determines the first order linearized dynamics of many complex processes on graphs and is therefore of particular importance.

4. More general graphs and networks

- For **multigraphs**, multiple edges between nodes and loops ($a_{ii} > 0$) are allowed.
- **Hypergraphs** (V, E) are generalizations in which an edge can connect any number of vertices. Formally, the set of **hyperedges** $E \subseteq \mathcal{P}(V)$ is a set of non-empty subsets of V.
- In **bipartite graphs** the edge set can be partitioned into two sets $V_1, V_2 \subseteq V$ each non-empty, with no connections within themselves, i.e. $a_{ij} = a_{ji} = 0$ for all $i, j \in V_1$ and also for all $i, j \in V_2$.
  Simple undirected examples include regular lattices $\mathbb{Z}^d$ for $d \geq 1$ which are partitioned into sites with even and odd parity. Feed-forward neural networks are examples of directed graphs with bipartite or multi-partite structure.
- **Multilayer networks** $M = (\mathbf{G}, \mathbf{C})$ consist of a family of m (weighted or unweighted) graphs $G_\alpha = (V_\alpha, E_\alpha)$ (called **layers** of M), and the set of interconnections between nodes of different layers

$$\mathbf{C} = \left\{ c_{\alpha,\beta} \subseteq V_\alpha \times V_\beta : \alpha, \beta \in \{1, \ldots, m\},\ \alpha \neq \beta \right\} .$$

  Real examples include transportation networks or social networks with different types of connections.

5. E-R Random graphs

Definition 5.1

An **(Erdős-Rényi, short E-R) random graph** $G \sim \mathcal{G}_{N,K}$ has uniform distribution on the set of all undirected graphs with N vertices and $K = |E|/2$ edges, i.e.

$$\mathbb{P}_{N,k}[G = (V,E)] = \delta_{|V|,N}\delta_{|E|/2,K} \Big/ \binom{N(N-1)/2}{K} .$$

An **(E-R) random graph** $G \sim \mathcal{G}_{N,p}$ has N vertices and each (undirected) edge is present independently with probability $p \in [0,1]$, i.e.

$$\mathbb{P}_{N,p}[G = (V,E)] = \delta_{|V|,N} p^{|E|/2}(1-p)^{N(N-1)/2-|E|/2} .$$

- The ensemble $\mathcal{G}_{N,p}$ is easier to work with and is mostly used in practice, and for N, K large, $\mathcal{G}_{N,K}$ is largely equivalent to $\mathcal{G}_{N,p}$ with $p = 2K/(N(N-1))$.
- Since edges are present independently, graphs $G \in \mathcal{G}_{N,p}$ should typically be uncorrelated. Indeed, one can show that $\chi(G)$, $\mathbb{E}[\chi] \to 0$ as $N \to \infty$.

5. E-R Random graphs - properties

- The number of undirected edges for $G \sim \mathcal{G}_{N,p}$ is **random**, $K \sim \mathrm{Bi}\left(\frac{N(N-1)}{2}, p\right)$.
  Thus, the average degree is **random**, $\langle k \rangle = 2K/N$ with $\mathbb{E}[\langle k \rangle] = (N-1)p$.
  For all i by homogeneity, $k_i \sim \mathrm{Bi}(N-1, p)$ and $\mathbb{E}[\langle k \rangle] = \mathbb{E}[k_i]$.
- The expected number of triangles in a $\mathcal{G}_{N,p}$ graph is $\binom{N}{3}p^3$,
  and the number of triples is $\binom{N}{3}3p^2$.
  Since fluctuations are of lower order, this implies for all $G_N \sim \mathcal{G}_{N,p}$

$$C(G_N) = \frac{3\binom{N}{3}p^3(1+o(1))}{\binom{N}{3}3p^2(1+o(1))} \to p \quad \text{as } N \to \infty \ .$$

- The **expected** degree distribution for $G_N \sim \mathcal{G}_{N,p}$ is binomial. In the limit $N \to \infty$ with $p = p_N = z/(N-1)$ keeping $z = \mathbb{E}[\langle k \rangle]$ fixed we have

$$\mathbb{E}[p(k)] = \mathbb{P}[k_i = k] = \binom{N-1}{k} p_N^k (1-p_N)^{N-1-k} \to \frac{z^k}{k!} e^{-z} \ .$$

  Therefore, E-R $\mathcal{G}_{N,p}$ graphs are sometimes called **Poisson random graphs**.
- In this scaling limit E-R graphs are **locally tree-like**, i.e. connected components

$$C_i^n := \{j \in V : j \leftrightarrow i, d_{ij} \leq n\} \ , \quad n \text{ fixed}$$

  are tree subgraphs as $N \to \infty$ with probability 1.
  Vertex degrees are $k_i \sim \mathrm{Poi}(z)$ and iid $k_j \sim \mathrm{Poi}(z) + 1$.

5. Percolation

Percolation studies robustness of connectivity properties of graphs under deletion of edges or vertices (e.g. random attacks or immunization).

Definition 5.2

Consider a connected, undirected graph $G = (V, E)$. **Bond percolation** is a static probabilistic model with state space

$$S = \Omega = \{0,1\}^E = \left\{ e_{ij} \in \{0,1\} : (i,j) \in E \right\} ,$$

and distribution $p = \mathbb{P}[e_{ij} = 1] = 1 - \mathbb{P}[e_{ij} = 0]$, i.e. $e_{ij} \sim \mathrm{Be}(p)$ iid with $p \in [0,1]$. Edges $(i,j) \in E$ are called **open** if $e_{ij} = 1$ and **closed** if $e_{ij} = 0$, and we denote by

$$G^o = (V, E^o) \quad \text{with} \quad E^o = \{(i,j) \in E : e_{ij} = 1\} \subseteq E$$

the (random) subgraph containing only open edges. A sequence of connected graphs G_N of increasing size $|V_N| = N$ exhibits **percolation with parameter** p if

$$|\bar{C}_N^o|/N \geq c > 0 \quad \text{as } N \to \infty \quad \text{with probability 1} ,$$

where $|\bar{C}_N^o| = \max_{i \in V_N} |C_i^o|$ is the size of the largest connected component $\bar{C}_N^o$ of G_N^o.

5. Percolation and E-R graphs

- Alternatively, percolation can be defined on an infinite graph G (e.g. $\mathbb{Z}^d$) with

$$\textbf{percolation probability} \quad \theta(p) = \mathbb{P}[|C_0| = \infty] \begin{cases} = 0\,, & \text{for } p < p_c \\ > 0\,, & \text{for } p > p_c \end{cases},$$

  changing behaviour at a **critical value** $p_c \in [0,1]$.
- In **site percolation** vertices and their adjacent edges are deleted.
- E-R random graphs $\mathcal{G}_{N,p}$ have the same distribution as open subgraphs $G_N^o \subseteq K_N$ under bond percolation with parameter p on the complete graph K_N.

Theorem 5.1 (Giant component for E-R graphs)

Consider $G_{N,p} \sim \mathcal{G}_{N,p}$ with $p = z/N$ and maximal connected component $\bar{C}_{N,p}$. Then

$$|\bar{C}_{N,p}| = \begin{cases} O(\log N)\,, & \text{for } z < 1 \\ O(N^{2/3})\,, & \text{for } z = 1 \\ O(N)\,, & \text{for } z > 1 \end{cases}.$$

$c(z) := \lim_{N\to\infty} |\bar{C}_{N,p}|/N$ is a continuous function of z. For $z > 1$, $\bar{C}_{N,p}$ is the only **giant component** of size $O(N)$, and the second largest is of order $O(\log N)$.

Local trees with $1 + \text{Poi}(z)$ degrees die out with probability 1 if and only if $z \leq 1$.

5. The Wigner semi-circle law

Theorem 5.2 (Wigner semi-circle law)

Let $A = (a_{ij} : i,j = 1,\ldots,N)$ be a real, symmetric matrix with iid entries a_{ij} for $i \leq j$ with finite moments, and $\mathbb{E}[a_{ij}] = 0$, $\mathrm{var}[a_{ij}] = \sigma^2$ (called a **Wigner matrix**). Then the spectral density ρ_N of the matrix $A/\sqrt{N}$ converges in distribution to

$$\rho_N(\lambda) \to \rho_{sc}(\lambda) := \begin{cases} (2\pi\sigma^2)^{-1}\sqrt{4\sigma^2 - \lambda^2}\,, & \text{if } |\lambda| < 2\sigma \\ 0\,, & \text{otherwise} \end{cases}.$$

- The bulk of eigenvalues of unscaled Wigner matrices typically lies in the interval $[-2\sqrt{N}\sigma, 2\sqrt{N}\sigma]$.
- Adjacency matrices A of $\mathcal{G}_{N,p}$ random graphs are symmetric with iid $\mathrm{Be}(p)$ entries with $\mathbb{E}[a_{ij}] = p$ and $\mathrm{var}[a_{ij}] = p(1-p)$, so are not Wigner matrices.
  A has a maximal **Perron-Frobenius eigenvalue** of order pN, but all other eigenvalues have modulus of order $\sqrt{N}$.
  For fixed $p > 0$ the Wigner semi-cirlce law holds for $N \to \infty$ as stated above.
  For scaled $p = p_N \gg p_c = 1/N$ the width of the support reduces to $4\sqrt{N}\sigma_N$ with $\sigma_N = \sqrt{p_N}$ and a modified version holds.
  For $p = p_N \ll p_c = 1/N$ the asymptotic spectral density deviates from ρ_{sc}.

5. Preferential attachment

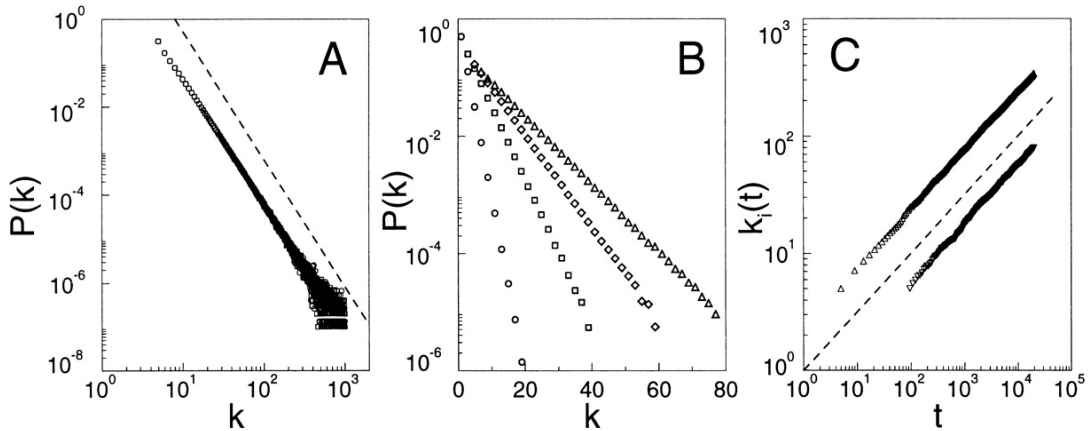
The prevalence of power-law degree distributions in real complex networks can be attributed to growth mechanisms subject to **preferential attachment**.

Definition 5.3

Starting with a complete graph (V_0, E_0) of $|V_0| = m_0$ nodes, at each time step $t = 1, \ldots, N - m_0$ a new node $j = t + m_0$ is added. It forms $m \leq m_0$ undirected edges with existing nodes $i \in V_{t-1}$ with a probability proportional to their degree $\pi_{j \leftrightarrow i} = k_i / \sum_{l \in V_t} k_l$ (preferential attachment).
The resulting, undirected graph with N nodes and $K = m_0(m_0 - 1)/2 + m(N - m_0)$ is called a **Barabási-Albert** random graph, denoted by $\mathcal{G}^{\mathrm{BA}}_{N,K}$.

- As $N \to \infty$, the average degree is $\langle k \rangle = 2m$ and the degree distribution $p_N(k)$ converges to a distribution $p(k)$ with **power law tail**, i.e. $p(k) = Ck^{-\alpha}$ as $k \to \infty$ where $\alpha = 3$, which is close to exponents observed for real-world networks. This is independent of the parameters m_0 and m.
- Characteristic path length and clustering coefficient typically behave like $L = O(\log N)$ and $C = O(N^{-0.75})$ for $\mathcal{G}^{\mathrm{BA}}_{N,K}$ graphs.
- The expected degree of nodes increases with their age.

5. Preferential attachment



(A) power law for $\gamma = 1, m_0 = m = 5, N = 200K$, (B) exponential tail for $\gamma = 0, m_0 = m = 1,\ 3,\ 5,\ 7,$
(C) degree increasing with time for $t_1 = 5,\ t_2 = 95$

taken from [A.-L. Barabási, R. Albert, Science **286**(5439), 509-512 (1999)]

- Variations of the model connecting to vertices i with probability proportional to $k_i + k_0$ lead to power law degree distributions with $\alpha = 3 + k_0/m$.
- For **non-linear preferentail attachment** proportional to k_i^γ the degree distribution converges to a stretched exponential tail $\exp(-Ck^{1-\gamma})$ for $\gamma \in [0, 1)$, and for $\gamma > 1$ a single vertex connects to almost all other vertices.
- There are many other and also older models that develop the idea of preferential attachment, for some of which one can do exact computations.
  (see e.g. [M.E.J. Newman, Siam Review **45**(2), 167256 (2003)]) .

5. Small-world networks

Definition 5.4

A sequence of connected graphs G_N with increasing size $|V_N| = N$ exhibits the **small-world property**, if the characteristic path length $L(G_N) = O(\log N)$.

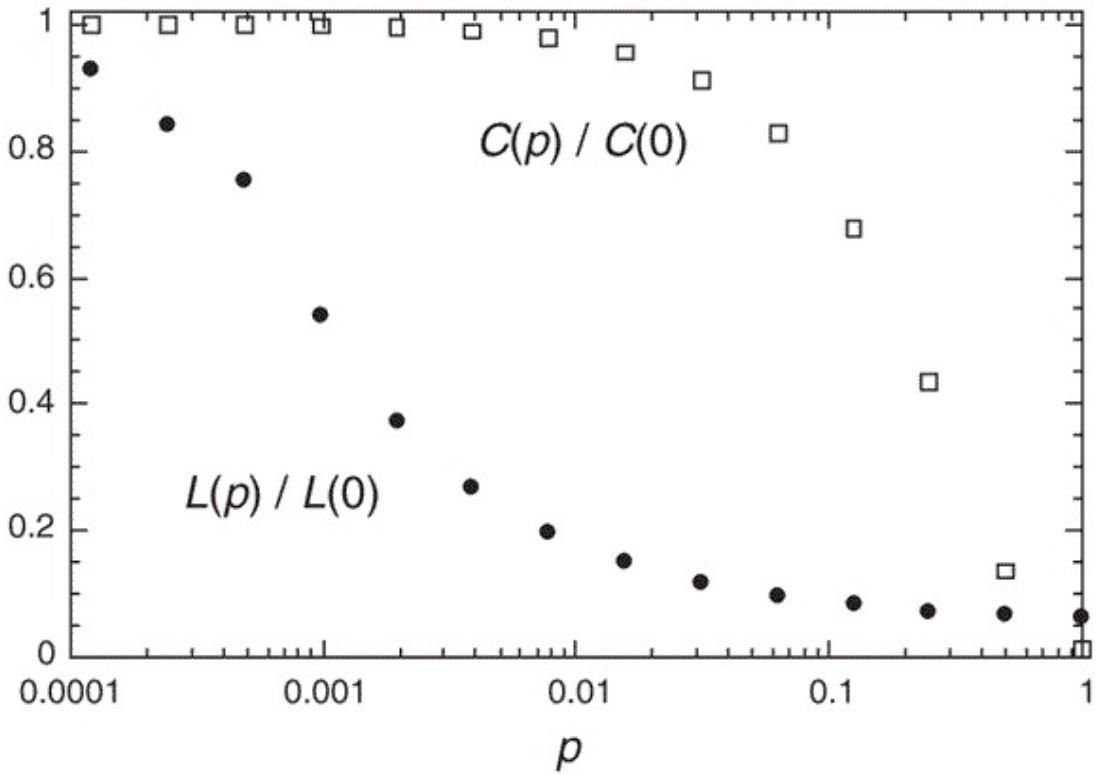
Examples include trees with degrees $k_i \geq 3$ and also the giant or largest component in E-R random graphs. In most graph models small-worldness is paired with low clustering coefficients, e.g. 0 for trees and p for $\mathcal{G}_{N,p}$ graphs. However, many real examples of small world networks exhibit also large clustering coefficients, such as networks of social contacts.

Definition 5.5

Consider a $2m$-regular ring graph with adjacency matrix $a_{ij} = \begin{cases} 1\,, & |i-j| \leq m \\ 0\,, & \text{otherwise} \end{cases}$ of size N with a total number of $K = mN$ undirected edges.
For all i, each edge (i,j) with a clockwise neighbour with $j > i$ is **rewired** with probability $p \in [0,1]$, i.e. replaced by an edge (i,l) where l is chosen uniformly among vertices not adjacent to i. The resulting graph is a **Watts-Strogatz random graph**, denoted by $\mathcal{G}^{\mathrm{WS}}_{N,K}$.

5. Watts-Strogatz model

- W-S random graphs interpolate between a regular lattice for $p = 0$ and a $\mathcal{G}_{N,K}$ E-R random graph conditioned on the event that all vertices have degree $k_i \geq m$.
- Expected clustering coefficient $\mathbb{E}[C(p)]$ and characteristic path length $\mathbb{E}[L(p)]$ are monotone decreasing functions of p and show the following behaviour.



$N = 1000$ and $m = 5$, taken from [D.J. Watts, S.H. Strogatz, Nature **393**, 440-442 (1998)]

5. Configuration model

Definition 5.6

The **configuration model** $\mathcal{G}_{N,D}^{\text{conf}}$ is defined as the uniform distribution among all undirected graphs with N vertices with a given degree sequence $D = (k_1, \ldots, k_N)$, such that $\sum_{i \in V} k_i = 2K$.

- Not all sequences D that sum to an even number are **graphical**.
- Sampling is usually done by attaching k_i half edges to each vertex i and matching them randomly. This can lead to self loops and rejections.
- General **randomized graphs** with given degree distribution $p(k)$ can be sampled in the same way. If $k_{\max} = \max_i k_i$ is bounded, one can show that these graphs exhibit a giant (connected) component of size $O(N)$ if

$$Q := \sum_{k \geq 0} k(k-2)p(k) > 0 \,,$$

  and if $Q < 0$ the largest component is of size $O(k_{\max}^2 \log N)$.
- For directed versions with D^{in} and D^{out} we need $\sum_{i \in V} k_i^{\text{in}} = \sum_{i \in V} k_i^{\text{out}}$.