Stochastic Modelling and Random Processes

Hand-out 5

Generating functions, branching processes

For a given sequence of numbers $a_0, a_1, \ldots \in \mathbb{R}$ we define the **generating function**

$$G(s) = \sum_{n=0}^{\infty} a_n\, s^n \ .$$

$s \geq 0$ is a dummy variable, and if the sequence is bounded the domain of definition of this power series includes the interval $[0, 1)$.

Examples.

- If $a_0 = a_1 = 1/2$ and $a_n = 0$ for $n \geq 2$, then $\quad G(s) = \frac{1}{2}(1+s)\ , \quad s \in [0, \infty)\ .$
- If $a_n = 2^{-n-1}$ then $\quad G(s) = \sum_{n=0}^{\infty} s^{-n-1}\, s^n = (2-s)^{-1}\ , \quad s \in [0, 2)\ .$

$G(s)$ is a convenient way of encoding the sequence, and often one can get an explicit formula.
Given a generating function $G(s)$, we can recover the sequence by differentiation

$$a_0 = G(0)\ , \quad a_1 = G'(0)\ , \quad a_2 = \frac{1}{2}\, G''(0)\ , \quad \ldots \quad a_n = \frac{1}{n!}\, G^{(n)}(0)\ .$$

We will often use genering functions to encode the sequence of probabilities $p_n = \mathbb{P}(X = n)$ of a non-negative, integer-valued random variable X,

$$G_X(s) = \sum_{n=0}^{\infty} p_n\, s^n = \mathbb{E}\big(s^X\big)\ , \quad s \in [0, 1]\ .$$

We call G_X also **probability generating function** of X, and

$$G_X(1) = 1\ , \quad G'_X(1) = \mathbb{E}(X) \quad \text{and} \quad \mathrm{Var}(X) = G''_X(1) + G'_X(1) - \big(G'_X(1)\big)^2\ .$$

Useful properties.

- If X, Y are independent non-negative, integer-valued random variables, then

  $$G_{X+Y}(s) = G_X(s)\, G_Y(s)\ .$$

  This is often much easier than evaluating the *convolution sum*

  $$\mathbb{P}(X + Y = n) = \sum_{k=0}^{n} \mathbb{P}(X = k)\, \mathbb{P}(Y = n - k)\ .$$

- More generally, if $X_1, X_2, \ldots$ are independent, identically distributed random variables (iidrv's), and N is a random number of summands, then

  $$Z = \sum_{k=1}^{N} X_k \quad \text{has generating function} \quad G_Z(s) = G_N\big(G_{X_1}(s)\big)\ .$$

A **branching process** $Z = (Z_n : n \in \mathbb{N})$ with state space $S = \mathbb{N}$ can be interpreted as a simple model for cell division or population growth. It is defined recursively by

$$Z_0 = 1\ , \quad Z_{n+1} = X_1^n + \ldots + X_{Z_n}^n \quad \text{for all } n \geq 0\ ,$$

where the $X_i^n \in \mathbb{N}$ are iidrv's denoting the offspring of individuum i in generation n. Z_n is then the size of the population in generation n.
Let $G(s) := \mathbb{E}\big(s^{X_1^0}\big)$ be the probability generating function of a single offspring X_1^0 and

$$G_n(s) := \mathbb{E}\big(s^{Z_n}\big) = \sum_{k=0}^{\infty} \mathbb{P}(Z_n = k)\, s^k\ .$$

Then we can derive the last formula on the previous page,

$$\begin{aligned} G_{n+1}(s) \ &= \ \mathbb{E}\big(s^{Z_{n+1}}\big) = \mathbb{E}\big(s^{X_1^n+\ldots+X_{Z_n}^n}\big) = \sum_{k=0}^{\infty} \mathbb{P}(Z_n = k)\, \mathbb{E}\big(s^{X_1^n+\ldots+X_k^n}\big) = \\ &= \ \sum_{k=0}^{\infty} \mathbb{P}(Z_n = k)\, \underbrace{\mathbb{E}\big(s^{X_1^n}\big)}_{=G(s)}{}^k = G_n\big(G(s)\big)\ . \end{aligned}$$

With average offspring $\mu := \mathbb{E}(X_1^0) = G'(0)$ we get with the chain rule and $G(1) = 1$,

$$\mathbb{E}(Z_{n+1}) = G'_{n+1}(1) = \Big(G_n\big(G(s)\big)\Big)'\Big|_{s=1} = G'_n\big(G(1)\big)\, G'(1) = \mathbb{E}(Z_n)\, \mu\ .$$

With the initial condition $Z_0 = 1$, this implies $\qquad \mathbb{E}(Z_n) = \mu^n \stackrel{n\to\infty}{\longrightarrow} \begin{cases} \infty & , \ \mu > 1 \\ 0 & , \ \mu < 1 \end{cases}\ .$

Probability of extinction.
$Z_n = 0$ is an absorbing state of the branching process corresponding to extinction of the population. Typically, the population either grows to infinite size or gets extinct in finite time. If T is the random time of extinction, we have

$$\mathbb{P}(T \leq n) = \mathbb{P}(Z_n = 0) = G_n(0)$$

for the probability that the population is extinct in generation n. Thus for the process to get extinct eventually (we call this event 'extinction') we have

$$\mathbb{P}(\text{extinction}) = \mathbb{P}(T < \infty) = \lim_{n\to\infty} \mathbb{P}(T \leq n) = \lim_{n\to\infty} G_n(0)\ .$$

So the event $T = \infty$ corresponds to 'non-extinction' or 'survival'.
Using a cobweb plot, one can easily see that this leads to

$$\mathbb{P}(\text{extinction}) = s^* \quad \text{where} \quad s^* = G(s^*)\ ,$$

is the smallest fixed point of G on $[0, 1]$.
The possible scenarios for the fate of the population are

$$\begin{aligned} &\mu \leq 1 \quad \Rightarrow \quad \mathbb{P}(\text{extinction}) = 1 \quad \text{and the population dies out for sure}\ , \\ &\mu > 1 \quad \Rightarrow \quad \mathbb{P}(\text{extinction}) < 1 \quad \text{and the population survives with positive probability}\ . \end{aligned}$$