Networks and Random Processes

Problem sheet 2

Sheet counts $50/150$ homework marks, [x] indicates weight of the question.
Please put solutions in my pigeon hole or give them to me by **Friday, 15.11.2019, 12pm noon**.

2.1 Kingman's coalescent [12]

Consider a system of L well mixed, coalescing particles. Each of the $\binom{L}{2}$ pairs of particles coalesces independently with rate 1. This can be interpreted as generating an ancestral tree of L individuals in a population model, tracing back to a single common ancestor.

(a) Let N_t be the number of particles at time t with $N_0 = L$. Give the transition rates of the process $(N_t : t \geq 0)$ on the state space $\{1, \ldots, L\}$, write down the generator $(\mathcal{L}f)(n)$ for $n \in \{1, \ldots, L\}$ and the master equation.
Is the process ergodic? Does it have absorbing states? Give all stationary distributions.

(b) Show that the mean time to absorption is given by $\mathbb{E}(T) = 2\big(1 - \frac{1}{L}\big)$.

(c) Write the generator of the rescaled process N_t/L and Taylor expand up to second order. Show that the slowed down, rescaled process $X_t^L := \frac{1}{L} N_{t/L} \to X_t$ converges to the process $(X_t : t \geq 0)$ with generator

$$\bar{\mathcal{L}} f(x) = -\frac{x^2}{2} f'(x) \quad \text{and state space} \quad (0, 1] \quad \text{with} \quad X_0 = 1 .$$

Convince yourself that this process is 'deterministic', i.e. $X_t = \mathbb{E}(X_t)$ for all $t \geq 0$, and compute X_t explicitly. How is your result compatible with the result from (b)?

(d) Generate sample paths of the process X_t^L for $L = 10, 100, 1000$ and compare to the solution X_t from (c) in a single plot.

2.2 Ornstein-Uhlenbeck process [10]

The Ornstein-Uhlenbeck process $(X_t : t \geq 0)$ is a diffusion process on $\mathbb{R}$ with generator

$$(\mathcal{L}f)(x) = -\alpha\, x\, f'(x) + \frac{1}{2}\sigma^2 f''(x) , \quad \alpha, \sigma^2 > 0 ,$$

and we consider a fixed initial condition $X_0 = x_0$.

(a) Use the evolution equation of expectation values of test functions $f : \mathbb{R} \to \mathbb{R}$

$$\frac{d}{dt}\mathbb{E}\big[f(X_t)\big] = \mathbb{E}\big[\mathcal{L}f(X_t)\big] ,$$

to derive ODEs for the mean $m(t) := \mathbb{E}[X_t]$ and the variance $v(t) := \mathbb{E}[X_t^2] - m(t)^2$.

(b) Solve the ODEs derived in (a). Using the fact that $(X_t : t \geq 0)$ is a Gaussian process, give the distribution of X_t for all $t \geq 0$.
What is the stationary distribution of the process?

(c) For $\alpha = 1$, $\sigma^2 = 1$ and $X_0 = 5$ simulate and plot a sample path of the process up to time $t = 10$, by numerically integrating the SDE with time steps $\Delta t = 0.1$ and 0.01.

2.3 Moran model and Wright-Fisher diffusion [16]

Consider a fixed population of L individuals. At time $t = 0$ each individuum i has a different type $X_0(i)$, for simplicity we simply put $X_0(i) = i$. In continuous time, each individuum independently, with rate 1, imposes its type on another randomly chosen individuum (or equivalently, kills it and puts its own offspring in its place).

(a) Give the state space of the Markov chain $(X_t : t \geq 0)$. Is it irreducible?
What are the stationary distributions?

(b) Let $N_t = \sum_{i=1}^{L} \delta_{X_t(i),k}$ be the number of individuals of a given type $k \in \{1, \ldots, L\}$ at time t, with $N_0 = 1$.
Is $(N_t : t \geq 0)$ a Markov process? Give the state space and the generator.
Is the process irreducible? What are the stationary distributions?
What is the limiting distribution as $t \to \infty$ for the initial condition $N_0 = 1$?

(c) From now on consider general initial conditions $N_0 = n \in \{0, \ldots L\}$.
Compute $m_1(t) := \mathbb{E}[N_t] \equiv n$ for all $t \geq 0$.
Compute $m_2(t) := \mathbb{E}[N_t^2]$. What happens in the limit $t \to \infty$?
Use this to compute the absorption probabilities as a function of the initial condition n.

(d) Consider the rescaled process $M_t^L := \frac{1}{L} N_{tL^\alpha}$ on the state space $[0, 1]$.
For which value of $\alpha > 0$ does M_t^L have a (non-trivial) scaling limit $(M_t : t \geq 0)$?
Compute the generator of this process and write down the Fokker-Planck equation.
(The scaling limit is called **Wright-Fisher diffusion**).

(e) For the limit process $(M_t : t \geq 0)$ in (d) compute $m(t) := \mathbb{E}[M_t]$ and $v(t) := \mathbb{E}[M_t^2] - m(t)^2$. Is it a Gaussian process?

2.4 Birth-death processes [12]

A birth-death process $(X_t : t \geq 0)$ is a continuous-time Markov chain with state space $S = \mathbb{N}_0 = \{0, 1, \ldots\}$ and jump rates

$$x \xrightarrow{\alpha_x} x + 1 \quad \text{for all } x \in S\ , \qquad x \xrightarrow{\beta_x} x - 1 \quad \text{for all } x \geq 1\ .$$

(a) Suppose $\alpha_x = \alpha > 0$ for $x \geq 0$ and $\beta_x = \beta > 0$ for $x > 0$.
Consider different cases depending on the choice of α and β where necessary:
- Is X irreducible? Give all communicating classes in $\mathbb{N}_0$ and state whether they are transient or null/positive recurrent.
- Give all stationary distributions and state whether they are reversible.
- Is the process ergodic?

(b) Suppose $\alpha_x = x\alpha$, $\beta_x = x\beta$ for $x \geq 0$ with $\alpha, \beta > 0$ and $X_0 = 1$.
Consider different cases depending on the choice of α and β where necessary:
- Is X irreducible? Give all communicating classes in $\mathbb{N}_0$ and state whether they are transient or null/positive recurrent.
- Give all stationary distributions and state whether they are reversible.
- Derive an equation for $\mu_t = \mathbb{E}[X_t]$ and solve it for initial condition $\mu_0 = 1$.
- Set up a recursion for the 'extinction probability' $h_x = \mathbb{P}[X_t \to 0 | X_0 = x]$ and give the smallest solution with boundary condition $h_0 = 1$.
- Is the process ergodic?