Stochastic Modelling and Random Processes

Problem sheet 3

Sheet counts 70/170 homework marks, [x] indicates weight of the question. Please put solutions in my pigeon hole or give them to me by Friday, 17.01.2020, 12pm noon.

3.1 Geometric Brownian motion

Let \((X_t : t \geq 0)\) be a Brownian motion with constant drift on \(\mathbb{R}\) with generator

\[
(\mathcal{L} f)(x) = \mu f'(x) + \frac{1}{2} \sigma^2 f''(x), \quad \mu \in \mathbb{R}, \sigma > 0,
\]

and initial condition \(X_0 = 0\). **Geometric Brownian motion** is defined as \((Y_t : t \geq 0)\) with \(Y_t = e^{X_t}\).

(a) Show that \((Y_t : t \geq 0)\) is a diffusion process on \([0, \infty)\) and compute its generator.

Write down the associated SDE and Fokker-Planck equation.

(b) Use the evolution equation of expectation values of test functions \(f : \mathbb{R} \to \mathbb{R}\)

\[
\frac{d}{dt} \mathbb{E}[f(Y_t)] = \mathbb{E}[\mathcal{L} f(Y_t)],
\]

to derive ODEs for the mean \(m(t) := \mathbb{E}[Y_t]\) and the second moment \(m_2(t) := \mathbb{E}[Y_t^2]\).

(No need to solve the ODEs).

(c) Under which conditions on \(\mu\) and \(\sigma^2\) is \((Y_t : t \geq 0)\) a martingale?

What is the asymptotic behaviour of the variance \(v(t) = m_2(t) - m(t)^2\) in that case?

(d) Show that \(\delta_0\) is the unique stationary distribution of the process on the state space \([0, \infty)\).

Under which conditions on \(\mu\) and \(\sigma^2\) does the process with \(Y_0 = 1\) converge to the stationary distribution?

Under which conditions on \(\mu\) and \(\sigma^2\) is the process ergodic? Justify your answer.

(e) For \(\sigma^2 = 1\) choose \(\mu = -1/2\) and two other values \(\mu < -1/2\) and \(\mu > -1/2\). Simulate and plot a sample path of the process with \(Y_0 = 1\) up to time \(t = 10\), by numerically integrating the corresponding SDE with time steps \(\Delta t = 0.1\) and 0.01.

3.2 Barabási-Albert model

Consider the Barabási-Albert model starting with \(m_0 = 5\) connected nodes, adding in each timestep a node linked to \(m = 5\) existing distinct nodes according to the preferential attachment rule. Simulate the model for \(N = |V| = 1000\), with at least 20 independent realizations.

(a) Plot the tail of the degree distribution in a double logarithmic plot for a single realization and for all 20, and compare to the power law with exponent \(-2\) (all in a single plot).

(b) Compute \(k_{nn}(k) = \mathbb{E}\left[ \sum_{i \in V} k_{nn,i} \delta_{k_i,k} / \sum_{i \in V} \delta_{k_i,k} \right] \) where \(k_{nn,i} = \frac{1}{k_i} \sum_{j \in V} a_{ij} k_j\), and decide whether the graphs are typically uncorrelated or (dis-)assortative.

(c) Plot the spectrum of the adjacency matrix \(A = (a_{ij})\) using all realizations with a kernel density estimate, and compare it to the Wigner semi-circle law with \(\sigma^2 = \text{var}[a_{ij}]\).
3.3 Erdős Rényi random graphs

Consider the Erdős Rényi random graph model and simulate at least 20 realizations of \( G_{N,p} \) graphs with \( p = p_N = z/N \), \( z = 0.1, 0.2, \ldots, 3.0 \) for \( N = 100 \) and \( N = 1000 \).

(a) Plot the expected size of the largest two components divided by \( N \), against \( z \) for both values of \( N \) in a single plot, using all 20 realizations with error bars indicating the standard deviation.

(b) For \( N = 1000 \) plot the (empirical approximation of the) expected local clustering coefficient \( \mathbb{E}[(C_i)] \) against \( z \) using all 20 realizations, including error bars indicating the standard deviation.

(c) For \( N = 1000 \) and your favourite value of \( z \in [0.5, 2] \), plot the degree distribution \( p(k) \) against \( k = 0, 1, \ldots \) using all 20 realizations, and compare it to the mass function of the Poisson distribution in a single plot.

(d) Consider \( z = 0.5, 1.5, 5 \) and 10. Plot the spectrum of the adjacency matrix \( A \) using all 20 realizations with a kernel density estimate, and compare it to the Wigner semi-circle law.

3.4 Contact process

Consider the CP \((\eta_t : t \geq 0)\) on the complete graph \( \Lambda = \{1, \ldots, L\} \) (i.e. \( q(i, j) = \lambda \) for all \( i \neq j \)) with state space \( S = \{0, 1\}^L \) and transition rates
\[
c(\eta, \eta') = \eta(i) + \lambda(1 - \eta(i)) \sum_{j \neq i} \eta(j),
\]
and generator given by \( (\mathcal{L} f)(\eta) = \sum_{i \in \Lambda} c(\eta, \eta')(f(\eta') - f(\eta)) \).

(a) Let \( N(\eta) := \sum_{i \in \Lambda} \eta(i) \in \{0, \ldots, L\} \) be the number of infected individuals in configuration \( \eta \). For any function \( f : \{0, \ldots, L\} \to \mathbb{R} \) show that we can write for the composed function \( f \circ N : S \to \mathbb{R} \)
\[
(\mathcal{L} f \circ N)(\eta) = \lambda(L - N)N[f(N + 1) - f(N)] + N[f(N - 1) - f(N)]
\]
for all \( \eta \in S \), where we use the simplified notation \( N = N(\eta) \) on the right-hand side.

Hint: Use \( N(\eta') = N(\eta) \pm 1 \) if \( \eta(i) = 0, 1 \), respectively, and \( (1 - \eta(i)) \eta(i) = 0 \).

Convince yourself that this implies that \((N_t : t \geq 0)\) with \( N_t := N(\eta_t) \) is a Markov chain on \( \{0, \ldots, L\} \) and write down its generator \( \mathcal{L} f(n) \).

(b) Is the process \((N_t : t \geq 0)\) irreducible, does it have absorbing states?

Give all stationary distributions. Is the process ergodic?

(c) Assume that \( \mathbb{E}(N_t^k) = \mathbb{E}(N_0)^k \) for all \( k \geq 1 \). This is called a mean-field assumption, meaning basically that we replace the random variable \( N_t \) by its expected value.

Use this assumption and the usual evolution equation as in Q2.1(b) to derive the mean-field rate equation for \( \rho(t) := \mathbb{E}(N_t)/L \),
\[
\frac{d}{dt} \rho(t) = h(\rho(t)) := -\rho(t) + L\lambda(1 - \rho(t))\rho(t).
\]

(d) Analyze this equation by finding the stable and unstable stationary points via \( h(\rho^*) = 0 \), and give the limiting behaviour of \( \rho(t) \) as \( t \to \infty \) depending on the parameter \( \lambda > 0 \).
3.5 Simulation of CP

Consider the contact process \((\eta_t : t \geq 0)\) as defined in Q2.4, but now on the one-dimensional lattice \(\Lambda_L = \{1, \ldots, L\}\) with connections only between nearest neighbours, i.e. \(q(i,j) = \delta_{j,i+1}\), and periodic boundary conditions.

The critical infection rate \(\lambda_c\) can be defined such that the infection on the infinite lattice \(\Lambda = \mathbb{Z}\) started from the fully infected lattice dies out for \(\lambda < \lambda_c\), and survives for \(\lambda > \lambda_c\). It is known numerically up to several digits, depends on the dimension, and is around 1.65 in our case.

All simulations of the process should be done with initial condition \(\eta_0(i) = 1\) for all \(i \in \Lambda\).

(a) Simulate the process for \(L = 128, 256, 512, 1024\) and parameters \(\lambda = 1.62, \ldots, 1.68\) with 0.01 increments (7 values) with at least 500 realizations each.

For each \(L\), plot the number of infected individuals \(N_t = \sum_{i \in \Lambda_L} \eta_t(i)\) averaged over realizations as a function of time up to time \(10 \times L\) for all values of \(\lambda\) as above in a single double-logarithmic plot.

Use the curvature of the plots to estimate \(\lambda_c(L)\).

Plot your estimates of \(\lambda_c(L)\) with error bars \(\pm 0.01\) against \(1/L\). Extrapolate to \(1/L \to 0\) to get an estimate of \(\lambda_c = \lambda_c(\infty)\) with a reasonable error bar.

This approach is called finite size scaling, in order to correct for systematic finite size effects which influence the critical value.

(b) Let \(T\) be the hitting time of the absorbing state \(\eta = 0\), i.e. the lifetime of the infection.

Measure the lifetime of the infection for \(\lambda = 1\) and \(\lambda = 1.8\) by running the process until extinction of the epidemic.

For \(\lambda = 1 < \lambda_c\) we expect \(T \propto C \log L + \text{small fluctuations}\) for some \(C > 0\). So use large system sizes e.g. \(L = 128, 256, 512, 1024\) (or larger), confirm that \(\mathbb{E}(T)\) scales like \(\log L\) and determine \(C\) by averaging at least 200 realizations of \(T\) for each \(L\).

Then shift your data \(T_i\) for each \(L\) by \(T_i - \mathbb{E}(T)\) and plot the ’empirical tail’ of the distribution of the shifted data, comparing to the Gumbel distribution (all in one plot with log-scale on the y-axis).

Look up the Gumbel distribution on Wikipedia, with mean 0 only one parameter needs fitting. Why could this be a good model for the noise here? Very short answer relating to extreme value statistics (see google) suffices.

For \(\lambda = 1.8 > \lambda_c\) we expect \(T \sim \text{Exp}(1/\mu)\) to be an exponential random variable with mean \(\mu = \mathbb{E}(T) \propto e^{CL}\) for some \(C > 0\). So use *small* system sizes e.g. \(L = 8, 10, 12, 14\) (see how far you can go), confirm that \(\mathbb{E}(T)\) scales like \(e^{CL}\) and determine \(C\) by averaging at least 200 realizations of \(T\).

Then rescale your data \(T_i\) for each \(L\) by \(T_i/\mathbb{E}(T)\) and plot the ’empirical tail’ of the distribution of the rescaled data, comparing to the theoretical tail \(e^{-t}\) (all in one plot with log-scale on the y-axis).

Recall: The empirical tail of data \(T = (T_1, \ldots, T_M)\) is the statistic
\[
\text{tail}_t(T) = \frac{1}{M} \sum_{i=1}^{M} \mathbb{1}_{T_i > t}.
\]

This decays from 1 to 0 as a (random) function of time \(t\).