Stochastic Modelling and Random Processes

Problem sheet 3

Sheet counts $50/150$ homework marks, [x] indicates weight of the question.
Please put solutions in my pigeon hole or give them to me by **Friday, 17.01.2020, 12pm noon**.

3.1 Geometric Brownian motion [15]

Let $(X_t : t \geq 0)$ be a Brownian motion with constant drift on $\mathbb{R}$ with generator

$$(\mathcal{L}f)(x) = \mu\, f'(x) + \frac{1}{2}\sigma^2\, f''(x)\ , \quad \mu \in \mathbb{R}, \sigma > 0\ ,$$

and initial condition $X_0 = 0$. **Geometric Brownian motion** is defined as

$$(Y_t : t \geq 0) \quad \text{with} \quad Y_t = e^{X_t}\ .$$

(a) Show that $(Y_t : t \geq 0)$ is a diffusion process on $[0, \infty)$ and compute its generator. Write down the associated SDE and Fokker-Planck equation.

(b) Use the evolution equation of expectation values of test functions $f : \mathbb{R} \to \mathbb{R}$

$$\frac{d}{dt}\mathbb{E}\big[f(Y_t)\big] = \mathbb{E}\big[\mathcal{L}f(Y_t)\big]\ ,$$

to derive ODEs for the mean $m(t) := \mathbb{E}[Y_t]$ and the second moment $m_2(t) := \mathbb{E}[Y_t^2]$. (No need to solve the ODEs).

(c) Under which conditions on μ and σ^2 is $(Y_t : t \geq 0)$ a martingale? What is the asymptotic behaviour of the variance $v(t) = m_2(t) - m(t)^2$ in that case?

(d) Show that δ_0 is the unique stationary distribution of the process on the state space $[0, \infty)$. Under which conditions on μ and σ^2 does the process with $Y_0 = 1$ converge to the stationary distribution?
Under which conditions on μ and σ^2 is the process ergodic? Justify your answer.

(e) For $\sigma^2 = 1$ choose $\mu = -1/2$ and two other values $\mu < -1/2$ and $\mu > -1/2$. Simulate and plot a sample path of the process with $Y_0 = 1$ up to time $t = 10$, by numerically integrating the corresponding SDE with time steps $\Delta t = 0.1$ and 0.01.

3.2 Barabási-Albert model [9]

Consider the Barabási-Albert model starting with $m_0 = 5$ connected nodes, adding in each timestep a node linked to $m = 5$ existing distinct nodes according to the preferential attachment rule. Simulate the model for $N = |V| = 1000$, with at least 20 independent realizations.

(a) Plot the tail of the degree distribution in a double logarithmic plot for a single realization and for all 20, and compare to the power law with exponent -2 (all in a single plot).

(b) Compute $k_{nn}(k) = \mathbb{E}\Big[\sum_{i\in V} k_{nn,i}\delta_{k_i,k}\Big/\sum_{i\in V}\delta_{k_i,k}\Big]$ where $k_{nn,i} = \frac{1}{k_i}\sum_{j\in V} a_{ij}k_j$, and decide whether the graphs are typically uncorrelated or (dis-)assortative.

(c) Plot the spectrum of the adjacency matrix $A = (a_{ij})$ using all realizations with a kernel density estimate, and compare it to the Wigner semi-circle law with $\sigma^2 = \mathrm{var}[a_{ij}]$.

3.3 Erdős Rényi random graphs [12]

Consider the Erdős Rényi random graph model and simulate at least 20 realizations of $\mathcal{G}_{N,p}$ graphs with $p = p_N = z/N$, $z = 0.1, 0.2, \ldots, 3.0$ for $N = 100$ and $N = 1000$.

(a) Plot the average size of the two largest components in each realization divided by N, against z for both values of N in a single plot (4 data series in total, use different colours). Use all 20 (or more) realizations and include error bars indicating the standard deviation.

(b) For $N = 1000$ plot the average local clustering coefficient $\langle C_i \rangle$ against z using all 20 realizations and $i = 1, \ldots, N$ for averaging, and including error bars indicating the standard deviation for all $20N$ data points.

(c) For $N = 1000$ and your favourite value of $z \in [0.5, 2]$, plot the degree distribution $p(k)$ against $k = 0, 1, \ldots$ using all 20 realizations, and compare it to the mass function of the $\mathrm{Poi}(z)$ Poisson distribution in a single plot.

(d) Consider $z = 0.5, 1.5, 5$ and 10. Plot the spectrum of the adjacency matrix A using all 20 realizations with a kernel density estimate, and compare it to the Wigner semi-circle law.

3.4 Contact process [14]

Consider the CP $(\eta_t : t \geq 0)$ on the complete graph $\Lambda = \{1, \ldots, L\}$ (i.e. $q(i,j) = \lambda$ for all $i \neq j$) with state space $S = \{0,1\}^L$ and transition rates

$$c(\eta, \eta^i) = \eta(i) + \lambda\big(1 - \eta(i)\big) \sum_{j \neq i} \eta(j) ,$$

and generator given by $\quad (\mathcal{L}f)(\eta) = \sum_{i \in \Lambda} c(\eta, \eta^i)\big(f(\eta^i) - f(\eta)\big)$.

(a) Let $N(\eta) := \sum_{i \in \Lambda} \eta(i) \in \{0, \ldots, L\}$ be the number of infected individuals in configuration η. For any function $f : \{0, \ldots, L\} \to \mathbb{R}$ show that we can write for the composed function $f \circ N : S \to \mathbb{R}$

$$(\mathcal{L}f \circ N)(\eta) = \lambda(L - N)N\big[f(N+1) - f(N)\big] + N\big[f(N-1) - f(N)\big]$$

for all $\eta \in S$, where we use the simplified notation $N = N(\eta)$ on the right-hand side.
Hint: Use $N(\eta^i) = N(\eta) \pm 1$ if $\eta(i) = 0, 1$, respectively, and $\big(1 - \eta(i)\big)\eta(i) = 0$.
Convince yourself that this implies that $(N_t : t \geq 0)$ with $N_t := N(\eta_t)$ is a Markov chain on $\{0, \ldots, L\}$ and write down its generator $\mathcal{L}f(n)$.

(b) Is the process $(N_t : t \geq 0)$ irreducible, does it have absorbing states?
Give all stationary distributions. Is the process ergodic?

(c) Assume that $\mathbb{E}\big(N_t^k\big) = \mathbb{E}(N_t)^k$ for all $k \geq 1$. This is called a **mean-field assumption**, meaning basically that we replace the random variable N_t by its expected value.
Use this assumption and the usual evolution equation as in Q2.1(b) to derive the **mean-field rate equation** for $\rho(t) := \mathbb{E}(N_t)/L$,

$$\frac{d}{dt}\rho(t) = h\big(\rho(t)\big) := -\rho(t) + L\lambda\big(1 - \rho(t)\big)\rho(t) .$$

(d) Analyze this equation by finding the stable and unstable stationary points via $h(\rho^*) = 0$, and give the limiting behaviour of $\rho(t)$ as $t \to \infty$ depending on the parameter $\lambda > 0$.

3.5 Simulation of CP* **not for credit**

Consider the contact process $(\eta_t : t \geq 0)$ as defined in Q2.4, but now on the one-dimensional lattice $\Lambda_L = \{1, \ldots, L\}$ with connections only between nearest neighbours, i.e. $q(i,j) = q(j,i) = \lambda\delta_{j,i+1}$, and periodic boundary conditions.
The critical infection rate λ_c can be defined such that the infection on the infinite lattice $\Lambda = \mathbb{Z}$ started from the fully infected lattice dies out for $\lambda < \lambda_c$, and survives for $\lambda > \lambda_c$. It is known numerically up to several digits, depends on the dimension, and is around 1.65 in our case.
All simulations of the process should be done with initial condition $\eta_0(i) = 1$ for all $i \in \Lambda$.

(a) Simulate the process for $L = 128, 256, 512, 1024$ and parameters $\lambda = 1.62, \ldots, 1.68$ with 0.01 increments (7 values) with at least 500 realizations each.
For each L, plot the number of infected individuals $N_t = \sum_{i\in\Lambda_L} \eta_t(i)$ averaged over realizations as a function of time up to time $10 \times L$ for all values of λ as above in a single double-logarithmic plot.
Use the curvature of the plots to estimate $\lambda_c(L)$.
Plot your estimates of $\lambda_c(L)$ with error bars ± 0.01 against $1/L$. Extrapolate to $1/L \to 0$ to get an estimate of $\lambda_c = \lambda_c(\infty)$ with a reasonable error bar.
This approach is called **finite size scaling**, in order to correct for systematic **finite size effects** which influence the critical value.

(b) Let T be the hitting time of the absorbing state $\eta = \mathbf{0}$, i.e. the lifetime of the infection. Measure the lifetime of the infection for $\lambda = 1$ and $\lambda = 1.8$ by running the process until extinction of the epidemic.

For $\lambda = 1 < \lambda_c$ we expect $T \propto C \log L$+small fluctuations for some $C > 0$. So use large system sizes e.g. $L = 128, 256, 512, 1024$ (or larger), confirm that $\mathbb{E}(T)$ scales like $\log L$ and determine C by averaging at least 200 realizations of T for each L.
Then shift your data T_i for each L by $T_i - \mathbb{E}(T)$ and plot the 'empirical tail' of the distribution of the shifted data, comparing to the **Gumbel distribution** (all in one plot with log-scale on the y-axis).
Look up the Gumbel distribution on Wikipedia, with mean 0 only one parameter needs fitting. Why could this be a good model for the noise here? Very short answer relating to **extreme value statistics** (see google) suffices.

For $\lambda = 1.8 > \lambda_c$ we expect $T \sim Exp(1/\mu)$ to be an exponential random variable with mean $\mu = \mathbb{E}(T) \propto e^{CL}$ for some $C > 0$. So use *small* system sizes e.g. $L = 8, 10, 12, 14$ (see how far you can go), confirm that $\mathbb{E}(T)$ scales like e^{CL} and determine C by averaging at least 200 realizations of T.
Then rescale your data T_i for each L by $T_i/\mathbb{E}(T)$ and plot the 'empirical tail' of the distribution of the rescaled data, comparing to the theoretical tail e^{-t} (all in one plot with log-scale on the y-axis).

Recall: The **empirical tail** of data $T = (T_1, \ldots, T_M)$ is the statistic
$tail_t(T) = \frac{1}{M}\sum_{i=1}^{M} \mathbb{1}_{T_i > t}$. This decays from 1 to 0 as a (random) function of time t.