Susana Gomes, Kamran Pentland 15.10.2021

Stochastic Modelling and Random Processes

Assignment 1

This assignment counts for $1/3$ of your homework marks. It is marked out of 100, and [x] indicates weight of each question.
Please submit your solutions via this link by **Monday, 25.10.2012, 12 noon UK time**.

All plots must contain axis labels and a legend (can be added by hand if necessary). Use your own judgement to find reasonable and relevant plot ranges.

1 Simple Random Walk [30]

Consider a Simple Random Walk with state space $S = \{1, \ldots, L\}$ with probabilities $p \in [0, 1]$ and $q = 1 - p$ to jump right and left:

$$p(x, y) = p\delta_{y,x+1} + q\delta y, x - 1.$$

(a) Consider periodic as well as closed boundary conditions:

- periodic boundary conditions, i.e. $p_{L,1} = p$, $p_{1,L} = q$,
- closed boundary conditions, i.e. $p_{1,1} = q$, $p_{L,L} = q$.

For each case, write down the transition matrix P of the process. Is the corresponding Markov chain irreducible? Give (all) the stationary distribution(s) $\boldsymbol{\pi}$ and state whether they are reversible. Where necessary, discuss the cases $p = 1$ and $p = q = 1/2$ separately from the general case $p \in (0, 1)$.

(b) Now consider absorbing boundary conditions ($p_{1,1} = p_{L,L} = 1$). Sketch the transition matrix P, decide whether the process is irreducible, and give all stationary distributions $\boldsymbol{\pi}$, stating whether they are reversible.
Let $h_k^L = \mathbb{P}[X_n = L \text{ for some } n \geq 0 | X_0 = k]$ be the absorption probability in site L. Give a recursion formula for h_k^L and solve it for $p \neq q$ and $p = q$.

(c) Simulate 500 realizations of a SRW with $L = 10$, closed boundary conditions and with a value for $p = 1 - q \in (0.6, 0.9)$ of your choice. For all simulations use $X_0 = 1$.
Plot the empirical distribution after 10 and 100 time steps in form of a histogram, and compare it with the theoretical stationary distribution from (a).

(d) Repeat a single realization of the same simulation up to 50 and 500 time steps and plot the fraction of time spent in each state as a histogram, comparing to the stationary distribution.

2 Generators and eigenvalues [30]

Consider the continuous-time Markov chain $(X_t : t \geq 0)$ with generator $G = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -4 & 3 \\ 0 & 1 & -1 \end{pmatrix}$.

(a) Draw a graph representation for the chain (i.e. connect the three states by their jump rates), and give the transition matrix P^Y of the corresponding jump chain $(Y_n : n \in \mathbb{N}_0)$.

(b) Consider the Taylor series of the matrix P_t and confirm that $\frac{d}{dt} P_t|_{t=0} = G$, $\frac{d^2}{dt^2} P_t|_{t=0} = G^2$ etc..
Assume that $G = Q^{-1} \Lambda Q$, with diagonal matrix $\Lambda \in \mathbb{C}^{3\times 3}$ and eigenvalues λ_i of G on the diagonal, and with some matrix $Q \in \mathbb{C}^{3\times 3}$. Show that

$$P(t) = \exp(tG) = Q^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} Q \, .$$

(You do not need to compute entries of the matrix Q)

(c) Compute λ_2 and λ_3. Use this to compute $p_{11}(t)$, i.e. determine the coefficients in

$$p_{11}(t) = a + b\, e^{\lambda_2 t} + c\, e^{\lambda_3 t} \, .$$

(Again, it is not necessary to compute the matrix Q, instead use what you know about $p_{11}(0)$ and $\frac{d}{dt} p_{11}(t)|_{t=0}$ etc.)

(d) What is the stationary distribution $\boldsymbol{\pi}$ of X?

3 Pólya urn models [40]

Consider the following experiment: Place k balls each of distinct color indexed by $i = 1, \dots, k$ in an urn. Draw one ball uniformly at random, then replace *two* balls of the colour just drawn in the urn. Iterate.

(a) Suppose we want to keep track of the contents using a stochastic process $\underline{X}(n)$ of the urn as a function of discrete time n. Give the state space S, the initial condition $\underline{X}(0)$ and the transition probabilities $p(\underline{x}, \underline{y})$, $\underline{x}, \underline{y} \in S$.

(b) For $k = 2$ sketch the state space and the transition probabilities between states and show that for all $(x_1, x_2) \in S$ and all $n \geq 1$

$$\mathbb{P}\big[\underline{X}(n) = (x_1, x_2)\big] = \frac{1}{n+1}\,\delta_{n+2, x_1+x_2}\ ,$$

i.e., the distribution at time n is uniform. Then use this to show that

$$\frac{1}{n+2}\underline{X}(n) \to (U, 1-U) \quad \text{as } n \to \infty,$$

where $U \sim U[0,1]$ is a uniform random variable on $[0,1]$.

Consider a **generalized Pólya urn model** with k types or colours on the same state space S as in (a) above. In this case, the transition probabilities are

$$p(\underline{x}, \underline{x} + \underline{e}_i) = \frac{f_i x_i^\gamma}{\sum_{j=1}^k f_j x_j^\gamma} \qquad \big(\text{and } p(\underline{x}, \underline{y}) = 0 \text{ if } \underline{y} \neq \underline{x} + \underline{e}_i\big)\ ,$$

where the $f_i > 0$ denote the **fitness** of type i and $\gamma \geq 0$ is a **reinforcement parameter**.

(c) Simulate the model for $k = 500$ types with equal fitness $f_i \equiv 1$ for $\gamma = 0,\ 0.5,\ 1$ and 1.5. For each γ, show the **empirical cumulative distribution functions** of $\underline{X}(n)$ for $n = 5000$, 20000 and 80000 in one plot, and do the same for the normalized data $\frac{1}{n+k}\underline{X}(n)$ (8 plots in total).

Choose the plot ranges reasonably and explain what you observe.

Background info: For $\gamma > 1$ it is known that the system exhibits **monopoly**, i.e. as $n \to \infty$ almost all balls in the urn will be of a single type.

(d) Play around with the fitness parameters f_i, find something 'interesting', and show one plot and write a few sentences to explain it.