Susana Gomes, Kamran Pentland 12.11.2021

Stochastic Modelling and Random Processes

Assignment 2

This assignment counts for $1/3$ of your homework marks. It is marked out of 100, and [x] indicates weight of each question.
Please submit your solutions via this link by **Monday, 22.11.2021, 12 noon UK time**.
All plots must contain axis labels and a legend (can be added by hand if necessary). Use your own judgement to find reasonable and relevant plot ranges.

1 SDEs and Gaussian processes [30]

Let B_t be a standard Brownian motion and $\mu(t) : \mathbb{R}^+ \mapsto \mathbb{R},\ \sigma(t) : \mathbb{R}^+ \mapsto \mathbb{R}$ be smooth functions of time, respectively. Define the stochastic process X_t to be the solution of the equation

$$dX_t = \mu(t)\, dt + \Sigma(t)\, dB_t, \quad X_0 = x,$$

where $x \in \mathbb{R}$ is a constant.

Hint: In what follows, you can use the following results about the stochastic integral without proof:

$$\mathbb{E}\left(\int_0^T f(t)\, dB_t\right) = 0 \qquad \textit{and}\ \mathbb{E}\left(\left(\int_0^T f(t)\, dB_t\right)^2\right) = \int_0^T f(t)^2\, dt$$

(a) Show that X_t is a Gaussian process with increments satisfying

$$X_t - X_s \sim \mathcal{N}\left(\int_s^t \mu(u)\, du,\ \int_s^t \Gamma(u)\, du\right),$$

where $\Gamma(t) = \sigma^2(t)$.

(b) Use the above result to develop an algorithm for simulating X_t for given $\mu(t),\ \sigma(t), x$.

(c) Use your algorithm to simulate $X_t,\ t \in [0, T]$ for $\mu(t) = 2t$, $\sigma(t) = 0.1\cos^2(t)$, and $x = 1$. Generate $N = 1000$ sample paths and calculate the mean and variance of this process as a function of time. Compare with the theoretical results.

(d) Solve the SDE numerically using the Euler-Maruyama scheme and compare the results with the theoretical results and (c).

2 Ornstein-Uhlenbeck process [30]

The Ornstein-Uhlenbeck process $(X_t : t \geq 0)$ is a diffusion process on $\mathbb{R}$ with generator

$$(\mathcal{L}f)(x) = -\alpha\, x\, f'(x) + \frac{1}{2}\sigma^2\, f''(x)\ , \quad \alpha, \sigma^2 > 0,$$

where we consider a fixed initial condition $X_0 = x_0$.

(a) Use the evolution equation of expectation values of test functions $f : \mathbb{R} \to \mathbb{R}$

$$\frac{d}{dt}\mathbb{E}\big[f(X_t)\big] = \mathbb{E}\big[\mathcal{L}f(X_t)\big],$$

to derive ODEs for the mean $m(t) := \mathbb{E}[X_t]$ and the variance $v(t) := \mathbb{E}[X_t^2] - m(t)^2$.
Hint: you can compare your results with the ones we obtained in lectures.

(b) Solve the ODEs derived in (a). Using the fact that $(X_t : t \geq 0)$ is a Gaussian process, give the distribution of X_t for all $t \geq 0$. What is the stationary distribution of the process?

(c) For $\alpha = 1$, $\sigma^2 = 1$ and $X_0 = 5$ simulate and plot a few sample paths of the process up to time $t = 10$, by numerically integrating the SDE with time steps $\Delta t = 0.1$ and 0.01. Compute the mean and variance as a function of time and compare your results with the theoretical results in (b).

3 Birth-death processes [40]

A birth-death process $(X_t : t \geq 0)$ is a continuous-time Markov chain with state space $S = \mathbb{N}_0 = \{0, 1, \ldots\}$ and jump rates

$$x \xrightarrow{\alpha_x} x + 1 \quad \text{for all } x \in S\ , \qquad x \xrightarrow{\beta_x} x - 1 \quad \text{for all } x \geq 1.$$

Hint: in what follows, you can search online for the conditions you need for the process to be recurrent, ergodic, etc.

(a) Suppose $\alpha_x = \alpha > 0$ for $x \geq 0$ and $\beta_x = \beta > 0$ for $x > 0$, and consider different cases depending on the choice of α and β where necessary.

   i. Is X irreducible? Give all communicating classes in $\mathbb{N}_0$ and state whether they are transient or null/positive recurrent.

   ii. Give all stationary distributions and state whether they are reversible.

   iii. Is the process ergodic?

   iv. Write down the generator G of this process and the master equation. Using this, write a differential equation for $\mu_t = \mathbb{E}(X_t)$ and solve it for $X_0 = 1$.

(b) Suppose $\alpha_x = x\alpha$, $\beta_x = x\beta$ for $x \geq 0$ with $\alpha, \beta > 0$ and $X_0 = 1$, and consider different cases depending on the choice of α and β where necessary.

   i. Is X irreducible? Give all communicating classes in $\mathbb{N}_0$ and state whether they are transient or null/positive recurrent.

   ii. Give all stationary distributions and state whether they are reversible.

   iii. Write down the generator G of this process and the master equation. Using this, write a differential equation for $\mu_t = \mathbb{E}(X_t)$ and solve it for $X_0 = 1$.

   iv. Set up a recursion for the 'extinction probability' $h_x = \mathbb{P}[X_t \to 0 | X_0 = x]$ and give the smallest solution with boundary condition $h_0 = 1$.

   v. Is the process ergodic?