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Stochastic Modelling and Random Processes

Assignment 3

This assignment counts for $1/3$ of your homework marks. It is marked out of 100, and [x] indicates weight of each question.
Please submit your solutions via this link by **Friday, 17.12.2021, 12 noon UK time**.
All plots must contain axis labels and a legend (can be added by hand if necessary). Use your own judgement to find reasonable and relevant plot ranges.

1 Erdős Rényi random graphs [40]

Consider the Erdős-Rényi random graph model and simulate at least 20 realisations of $\mathcal{G}_{N,p}$ graphs with $p = p_N = z/N$, for $z = 0.1, 0.2, \ldots, 3.0$ for $N = 100$ and $N = 1000$.

(a) Plot the average size of the two largest components in each realisation divided by N, against z for both values of N in a single plot (4 data series in total, use different colours). Use all 20 (or more) realisations and include error bars indicating the standard deviation.

(b) For $N = 1000$, plot the average local clustering coefficient $\langle C_i \rangle$ against z using all 20 realisations and $i = 1, \ldots, N$ for averaging, and including error bars indicating the standard deviation for all $20N$ data points.

(c) Use results in lectures to state what is the expected number of edges, as well as the expected average degree as a function of z and N and plot these as a function of z for the two values of N, comparing your results with the expected ones. Include error bars to indicate the standard deviation for your data points.

(d) For $N = 1000$ and your favourite value of $z \in [0.5, 2]$, plot the degree distribution $p(k)$ against $k = 0, 1, \ldots$ using all 20 realisations, and compare it to the mass function of the Poi(z) Poisson distribution in a single plot.

(e) Consider $z = 0.5, 1.5, 5$ and 10. Plot the spectrum of the adjacency matrix A using all 20 realisations with a kernel density estimate, and compare it to the Wigner semi-circle law. Comment on your results based on what you expect from the lectures.

2 Barabási-Albert model [30]

Consider the Barabási-Albert model starting with $m_0 = 5$ connected nodes, adding in each timestep a node linked to $m = 5$ existing distinct nodes according to the preferential attachment rule. Simulate the model for $N = |V| = 1000$, with at least 20 independent realisations.

(a) Plot the tail of the degree distribution in a double logarithmic plot for a single realisation and for the average of all 20 realisations, and compare to the power law with exponent -2 (all in a single plot).

(b) Compute the nearest neighbour

$$k_{nn}(k) = \mathbb{E}\Big[\sum_{i\in V} k_{nn,i}\delta_{k_i,k} \Big/ \sum_{i\in V} \delta_{k_i,k}\Big], \quad \text{where} \quad k_{nn,i} = \frac{1}{k_i}\sum_{j\in V} a_{ij}k_j,$$

and decide whether the graphs are typically uncorrelated or (dis-)assortative.

(c) Plot the spectrum of the adjacency matrix $A = (a_{ij})$ using all realisations with a kernel density estimate, and compare it to the Wigner semi-circle law with $\sigma^2 = \mathrm{var}[a_{ij}]$. Comment on your results.

3 Dorogovtsev-Mendes-Samukhin model [30]

Consider the following generalisation of the Barabási-Albert model: As before, start with $m_0 = 5$ connected nodes, but in each timestep add a node j, linked to $m = 5$ existing distinct nodes according to the new probability (to be adapted to avoid double edges)

$$\pi_{j\leftrightarrow i} = \frac{k_0 + k_i}{\sum_{i\in V(t)}(k_i + k_0)}\ , \quad k_0 \in \mathbb{N}_0\ .$$

Simulate the model for three different values of $k_0 = 1, 2, 4$ to generate graphs of size $N = |V| = 1000$, with 20 independent realisations in each case.

(a) Plot the tail of the degree distribution in a double logarithmic plot for a single realisation and for the average of all 20 realisations. For each k_0 compare the tail to the power law with exponent $-2 - k_0/m$.

(b) Compute the nearest neighbour

$$k_{nn}(k) = \mathbb{E}\Big[\sum_{i\in V} k_{nn,i}\delta_{k_i,k} \Big/ \sum_{i\in V} \delta_{k_i,k}\Big], \quad \text{where} \quad k_{nn,i} = \frac{1}{k_i}\sum_{j\in V} a_{ij}k_j,$$

and decide whether the graphs are typically uncorrelated or assortative.

(c) Plot the spectrum of the adjacency matrix A using all realisations with a kernel density estimate, and compare it to the Wigner semi-circle law as before. Comment on your results.