Stochastic Modelling and Random Processes

Assignment 2

This assignment counts for $1/3$ of your homework marks. It is marked out of 100, and [x] indicates the weight of each question.

You need to **justify all your answers** unless it is clear you do not need to. All plots must contain axis labels and a legend (use your own judgement to find reasonable and relevant plot ranges), and corresponding comments on the text of your answers.

The written part of your assignment should be submitted as a pdf file (either latex, scan of handwritten answers, or extract a pdf version of a Jupyter notebook or similar) and you should also include your code (Jupyter notebook, .m file, or any other format you choose to use). Your files should be named MA933_assignment2_1234567 (where 1234567 is replaced by your university ID number).

You should submit your solutions via this link by **Tuesday, 28.11.2022, 5pm UK time**.

1 SDEs and Gaussian processes [30]

Let B_t be a standard Brownian motion and $\mu(t) : \mathbb{R}^+ \mapsto \mathbb{R},\ \sigma(t) : \mathbb{R}^+ \mapsto \mathbb{R}$ be smooth functions of time, respectively. Define the stochastic process X_t to be the solution of the equation

$$dX_t = \mu(t)\, dt + \Sigma(t)\, dB_t, \quad X_0 = x,$$

where $x \in \mathbb{R}$ is a constant.

Hint: In what follows, you can use the following results about the stochastic integral without proof:

$$\mathbb{E}\left(\int_0^T f(t)\, dB_t\right) = 0, \qquad \textit{and}\ \mathbb{E}\left(\left(\int_0^T f(t)\, dB_t\right)^2\right) = \int_0^T f(t)^2\, dt.$$

(a) Show that X_t is a Gaussian process with increments satisfying

$$X_t - X_s \sim \mathcal{N}\left(\int_s^t \mu(u)\, du,\ \int_s^t \Gamma(u)\, du\right),$$

where $\Gamma(t) = \sigma^2(t)$.

(b) Use the above result to develop an algorithm for simulating X_t for given $\mu(t),\ \sigma(t), x$.

(c) Use your algorithm to simulate $X_t,\ t \in [0, T]$ for $\mu(t) = 2t,\ \sigma(t) = 0.1\cos^2(t)$, and $x = 1$. Generate $N = 1000$ sample paths and calculate the mean and variance of this process as a function of time. Compare with the theoretical results.

(d) Solve the SDE numerically using the Euler-Maruyama scheme and compare the results with the theoretical results and (c).

2 Ornstein-Uhlenbeck process [30]

The Ornstein-Uhlenbeck process $(X_t : t \geq 0)$ is a diffusion process on $\mathbb{R}$ with generator

$$(\mathcal{L}f)(x) = -\alpha\, x\, f'(x) + \frac{1}{2}\sigma^2\, f''(x)\;, \quad \alpha, \sigma^2 > 0,$$

where we consider a fixed initial condition $X_0 = x_0$.

(a) Use the evolution equation of expectation values of test functions $f : \mathbb{R} \to \mathbb{R}$

$$\frac{d}{dt}\mathbb{E}\big[f(X_t)\big] = \mathbb{E}\big[\mathcal{L}f(X_t)\big],$$

to derive ODEs for the mean $m(t) := \mathbb{E}[X_t]$ and the variance $v(t) := \mathbb{E}[X_t^2] - m(t)^2$.
Hint: you can compare your results with the ones we obtained in lectures.

(b) Solve the ODEs derived in (a). Using the fact that $(X_t : t \geq 0)$ is a Gaussian process, give the distribution of X_t for all $t \geq 0$. What is the stationary distribution of the process?

(c) For $\alpha = 1$, $\sigma^2 = 1$ and $X_0 = 5$ simulate and plot a few sample paths of the process up to time $t = 10$, by numerically integrating the SDE with time steps $\Delta t = 0.1$ and 0.01. Compute the mean and variance as a function of time and compare your results with the theoretical results in (b).

3 Birth-death processes [40]

A birth-death process $(X_t : t \geq 0)$ is a continuous-time Markov chain with state space $S = \mathbb{N}_0 = \{0, 1, \ldots\}$ and jump rates

$$x \xrightarrow{\alpha_x} x+1 \quad \text{for all } x \in S\;, \qquad x \xrightarrow{\beta_x} x-1 \quad \text{for all } x \geq 1.$$

Hint: in what follows, you can search online for the conditions you need for the process to be recurrent, ergodic, etc. You can use, for example, the Wikipedia page for birth-death processes, or references therin, in particular [2].

(a) Suppose $\alpha_x = \alpha > 0$ for $x \geq 0$ and $\beta_x = \beta > 0$ for $x > 0$, and consider different cases depending on the choice of α and β where necessary.

  i. Is X irreducible? Give all communicating classes in $\mathbb{N}_0$ and state whether they are transient or null/positive recurrent.
  ii. Give all stationary distributions and state whether they are reversible.
  iii. Is the process ergodic?
  iv. Write down the generator G of this process and the master equation. Using this, write a differential equation for $\mu_t = \mathbb{E}(X_t)$ and solve it for $X_0 = 1$.

(b) Suppose $\alpha_x = x\alpha$, $\beta_x = x\beta$ for $x \geq 0$ with $\alpha, \beta > 0$ and $X_0 = 1$, and consider different cases depending on the choice of α and β where necessary.

  i. Is X irreducible? Give all communicating classes in $\mathbb{N}_0$ and state whether they are transient or null/positive recurrent.
  ii. Give all stationary distributions and state whether they are reversible.
  iii. Write down the generator G of this process and the master equation. Using this, write a differential equation for $\mu_t = \mathbb{E}(X_t)$ and solve it for $X_0 = 1$.
  iv. Set up a recursion for the 'extinction probability' $h_x = \mathbb{P}[X_t \to 0 | X_0 = x]$ and give the smallest solution with boundary condition $h_0 = 1$.
  v. Is the process ergodic?