MA933 - Stochastic Modelling and Random Processes

MSc in Mathematics of Systems

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These notes and other information about the course are available on
www2.warwick.ac.uk/fac/sci/mathsys/courses/msc/ma933/
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References

- C.W. Gardiner: Handbook of Stochastic Methods (3rd edition), Springer 2004
- G. Grimmett: Probability on Graphs, CUP 2010
  http://www.statslab.cam.ac.uk/~grg/books/pgs.html
1. Probability

- **sample space** $\Omega$  \quad (e.g. $\{H, T\}$, $\{H, T\}^N$, {paths of a stochastic process})
- **events** $A \subseteq \Omega$ \quad (measurable) subsets \quad (e.g. odd numbers on a die)
  \quad $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is the set of all events (subset of the powerset)

**Definition 1.1**

A **probability distribution** $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ which is

(i) normalized, i.e. $\mathbb{P}[\emptyset] = 0$ \quad and \quad $\mathbb{P}[\Omega] = 1$

(ii) additive, i.e. $\mathbb{P}[\bigcup_i A_i] = \sum_i \mathbb{P}[A_i]$, 
  where $A_1, A_2, \ldots$ is a collection of disjoint events, i.e. $A_i \cap A_j = \emptyset$ for all $i, j$.

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **probability space**.

- For **discrete** $\Omega$: \quad $\mathcal{F} = \mathcal{P}(\Omega)$ \quad and \quad $\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega]$  
  e.g. $\mathbb{P}[\text{even number on a die}] = \mathbb{P}[2] + \mathbb{P}[4] + \mathbb{P}[6] = \frac{1}{2}$

- For **continuous** $\Omega$ \quad (e.g. $[0, 1]$): \quad $\mathcal{F} \subsetneq \mathcal{P}(\Omega)$
1. Independence and conditional probability

- Two events $A, B \subseteq \Omega$ are called independent if $P[A \cap B] = P[A]P[B]$.

**Example.** Rolling a die repeatedly

- If $P[B] > 0$ then the conditional probability of $A$ given $B$ is

$$P[A|B] := \frac{P[A \cap B]}{P[B]}.$$

If $A$ and $B$ are independent, then $P[A|B] = P[A]$.

**Lemma 1.1 (Law of total probability)**

Let $B_1, \ldots, B_n$ be a partition of $\Omega$ such that $P[B_i] > 0$ for all $i$. Then

$$P[A] = \sum_{i=1}^{n} P[A \cap B_i] = \sum_{i=1}^{n} P[A|B_i]P[B_i].$$

Note that also $P[A|C] = \sum_{i=1}^{n} P[A|C \cap B_i]P[B_i|C]$ provided $P[C] > 0$. 
1. Random variables

**Definition 1.2**

A **random variable** $X$ is a (measurable) function $X : \Omega \to \mathbb{R}$.

The **distribution function** of the random variable is

$$F(x) = \mathbb{P}[X \leq x] = \mathbb{P}[\{\omega : X(\omega) \leq x\}] .$$

$X$ is called **discrete**, if it only takes values in a countable subset $\Delta = \{x_1, x_2, \ldots\} \subseteq \mathbb{R}$, and its distribution is characterized by the **probability mass function**

$$\pi(x) := \mathbb{P}[X = x], \quad x \in \Delta .$$

$X$ is called **continuous**, if its distribution function is

$$F(x) = \int_{-\infty}^{x} f(y) \, dy \quad \text{for all } x \in \mathbb{R} ,$$

where $f : \mathbb{R} \to [0, \infty)$ is the **probability density function** (PDF) of $X$. 
1. Random variables

- In general, \( f = F' \) is given by the derivative (exists for cont. rv’s).
  For discrete rv’s, \( F \) is a step function with ’PDF’
  \[
  f(x) = F'(x) = \sum_{y \in \Delta} \pi(y) \delta(x - y).
  \]

- The **expected value** of \( X \) is given by \[
  \mathbb{E}[X] = \left\{ \begin{array}{ll}
  \sum_{x \in \Delta} x \pi(x) \\
  \int_{\mathbb{R}} x f(x) \, dx
  \end{array} \right.
  \]

- The **variance** is given by \[
  \text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2,
  \]
  the **covariance** of two r.v.s by \[
  \text{Cov}[X, Y] := \mathbb{E}[XY] - \mathbb{E}[X] \mathbb{E}[Y].
  \]

- Two random variables \( X, Y \) are independent if the events \( \{X \leq x\} \) and \( \{Y \leq y\} \)
  are independent for all \( x, y \in \mathbb{R} \). This implies for **joint distributions**
  \[
  f(x, y) = f^X(x) f^Y(y) \quad \text{or} \quad \pi(x, y) = \pi^X(x) \pi^Y(y)
  \]
  with **marginals** \[
  f^X(x) = \int_{\mathbb{R}} f(x, y) \, dy \quad \text{and} \quad \pi^X(x) = \sum_{y \in \Delta_y} \pi(x, y).
  \]

- Independence implies \( \text{Cov}[X, Y] = 0 \), i.e. \( X \) and \( Y \) are **uncorrelated**.
  The inverse is in general false, but holds if \( X \) and \( Y \) are Gaussian.
1. Simple random walk

Definition 1.3

Let $X_1, X_2, \ldots \in \{-1, 1\}$ be a sequence of independent, identically distributed random variables (iidrv’s) with

$$p = \mathbb{P}[X_i = 1] \quad \text{and} \quad q = \mathbb{P}[X_i = -1] = 1 - p.$$  

The sequence $Y_0, Y_1, \ldots$ defined as $Y_0 = 0$ and $Y_n = \sum_{k=1}^{n} X_k$ is called the simple random walk (SRW) on $\mathbb{Z}$.

- for a single increment $X_k$ we have
  
  $$\mathbb{E}[X_k] = p - q = 2p - 1, \quad \text{var}[X_k] = p + q - (p - q)^2 = 4p(1 - p)$$

- $\mathbb{E}[Y_n] = \mathbb{E} \left[ \sum_{k=1}^{n} X_k \right] = \sum_{k=1}^{n} \mathbb{E}[X_k] = n(2p - 1)$
  
  (expectation is a linear operation)

- $\text{var}[Y_n] = \text{var} \left[ \sum_{k=1}^{n} X_k \right] = \sum_{k=1}^{n} \text{var}[X_k] = 4np(1 - p)$
  
  (for a sum of independent rv’s the variance is additive)
1. LLN and CLT

**Theorem 1.2 (Weak law of large numbers (LLN))**

Let $X_1, X_2, \ldots \in \mathbb{R}$ be a sequence of iidrv’s with $\mu := \mathbb{E}[X_k] < \infty$ and $\mathbb{E}[|X_k|] < \infty$. Then

$$\frac{1}{n} Y_n = \frac{1}{n} \sum_{k=1}^{n} X_k \to \mu \quad \text{as} \quad n \to \infty$$

in distribution (i.e. the distr. fct. of $Y_n$ converges to $1_{[\mu, \infty)}(x)$ for $x \neq \mu$).

**Theorem 1.3 (Central limit theorem (CLT))**

Let $X_1, X_2, \ldots \in \mathbb{R}$ be a sequence of iidrv’s with $\mu := \mathbb{E}[X_k] < \infty$ and $\sigma^2 := \text{var}[X_k] < \infty$. Then

$$\frac{Y_n - n\mu}{\sigma \sqrt{n}} = \frac{1}{\sigma \sqrt{n}} \sum_{k=1}^{n} (X_k - \mu) \to \xi \quad \text{as} \quad n \to \infty$$

in distr., where $\xi \sim N(0, 1)$ is a **standard Gaussian** with PDF $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

**Expansion.** as $n \to \infty$, $\sum_{k=1}^{n} X_k = n\mu + \sqrt{n} \sigma \xi + o(\sqrt{n})$, $\xi \sim N(0, 1)$
1. Discrete-time Markov processes

**Definition 1.4**

A **discrete-time stochastic process** with state space \( S \) is a sequence \( Y_0, Y_1, \ldots = (Y_n : n \in \mathbb{N}_0) \) of random variables taking values in \( S \).

The process is called **Markov**, if for all \( A \subseteq S, n \in \mathbb{N}_0 \) and \( s_0, \ldots, s_n \in S \)

\[
P(Y_{n+1} \in A|Y_n = s_n, \ldots, Y_0 = s_0) = P(Y_{n+1} \in A|Y_n = s_n) .
\]

A Markov process (MP) is called **homogeneous** if for all \( A \subseteq S, n \in \mathbb{N}_0 \) and \( s \in S \)

\[
P(Y_{n+1} \in A|Y_n = s) = P(Y_1 \in A|Y_0 = s) .
\]

If \( S \) is discrete, the MP is called a **Markov chain (MC)**.

The generic probability space \( \Omega \) is the **path space**

\[
\Omega = D(\mathbb{N}_0, S) := S^{\mathbb{N}_0} = S \times S \times \ldots
\]

which is uncountable even when \( S \) is finite. For a given \( \omega \in \Omega \) the function \( n \mapsto Y_n(\omega) \) is called a **sample path**.

Up to finite time \( N \) and with finite \( S \), \( \Omega_N = S^{N+1} \) is finite.
1. Discrete-time Markov processes

Examples.

- For the simple random walk we have state space $S = \mathbb{Z}$ and $Y_0 = 0$. Up to time $N$, $\mathbb{P}$ is a distribution on the finite path space $\Omega_N$ with

$$\mathbb{P}(\omega) = \begin{cases} p \# \text{ of up-steps} & q \# \text{ of down-steps} \\ 0 & \text{path } \omega \text{ not possible} \end{cases}, \text{ path } \omega \text{ possible}$$

There are only $2^N$ paths in $\Omega_N$ with non-zero probability. For $p = q = 1/2$ they all have the same probability $(1/2)^N$.

- For the generalized random walk with $Y_0 = 0$ and increments $Y_{n+1} - Y_n \in \mathbb{R}$, we have $S = \mathbb{R}$ and $\Omega_N = \mathbb{R}^N$ with an uncountable number of possible paths.

- A sequence $Y_0, Y_1, \ldots \in S$ of iidrv’s is also a Markov process with state space $S$.

- Let $S = \{1, \ldots, 52\}$ be a deck of cards, and $Y_1, \ldots, Y_{52}$ be the cards drawn at random without replacement. Is this a Markov process?
1. Discrete-time Markov chains

Proposition 1.4

Let \((X_n : n \in \mathbb{N}_0)\) by a homogeneous DTMC with discrete state space \(S\). Then the transition function

\[ p_n(x, y) := \mathbb{P}[X_n = y|X_0 = x] = \mathbb{P}[X_{k+n} = y|X_k = x] \quad \text{for all } k \geq 0 \]

is well defined and fulfills the Chapman Kolmogorov equations

\[ p_{k+n}(x, y) = \sum_{z \in S} p_k(x, z) p_n(z, y) \quad \text{for all } k, n \geq 0, \ x, y \in S. \]

Proof. We use the law of total probability, the Markov property and homogeneity

\[
\begin{align*}
\mathbb{P}[X_{k+n} = y|X_0 = x] &= \sum_{z \in S} \mathbb{P}[X_{k+n} = y|X_k = z, X_0 = x] \mathbb{P}[X_k = z|X_0 = x] \\
&= \sum_{z \in S} \mathbb{P}[X_{k+n} = y|X_k = z] \mathbb{P}[X_k = z|X_0 = x] \\
&= \sum_{z \in S} \mathbb{P}[X_n = y|X_0 = z] \mathbb{P}[X_k = z|X_0 = x]
\end{align*}
\]
1. Markov chains

- In matrix form with \( P_n = (p_n(x, y) : x, y \in S) \) the Chapman Kolmogorov equations read
  \[
P_{n+k} = P_n P_k \quad \text{and in particular} \quad P_{n+1} = P_n P_1.
  \]
  With \( P_0 = I \), the obvious solution to this recursion is
  \[
P_n = P^n \quad \text{where we write} \quad P_1 = P = (p(x, y) : x, y \in S).
  \]

- The transition matrix \( P \) and the initial condition \( X_0 \in S \) completely determine a homogeneous DTMC, since for all \( k \geq 1 \) and all events \( A_1, \ldots, A_k \subseteq S \)
  \[
  \mathbb{P}[X_1 \in A_1, \ldots, X_k \in A_k] = \sum_{s_1 \in A_1} \cdots \sum_{s_k \in A_k} p(X_0, s_1)p(s_1, s_2) \cdots p(s_{k-1}, s_k).
  \]

- Fixed \( X_0 \) can be replaced by an initial distribution \( \pi_0(x) := \mathbb{P}[X_0 = x] \).
  The distribution at time \( n \) is then
  \[
  \pi_n(x) = \sum_{y \in S} \sum_{s_1 \in S} \cdots \sum_{s_{n-1} \in S} \pi_0(y)p(y, s_1) \cdots p(s_{n-1}, x) \quad \text{or} \quad \langle \pi_n \rangle = \langle \pi_0 \rangle P^n.
  \]
1. Transition matrices

The transition matrix $P$ is **stochastic**, i.e.

$$p(x, y) \in [0, 1] \quad \text{and} \quad \sum_y p(x, y) = 1 ,$$

or equivalently, the column vector $|1\rangle = (1, \ldots, 1)^T$ is **eigenvector** with **eigenvalue** $1$: $P|1\rangle = |1\rangle$

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**Example 1 (Random walk with boundaries)**

Let $(X_n : n \in \mathbb{N}_0)$ be a SRW on $S = \{1, \ldots, L\}$ with $p(x, y) = p\delta_{y,x+1} + q\delta_{y,x-1}$.

The boundary conditions are

- **periodic** if $p(L, 1) = p$, $p(1, L) = q$,
- **absorbing** if $p(L, L) = 1$, $p(1, 1) = 1$,
- **closed** if $p(1, 1) = q$, $p(L, L) = p$,
- **reflecting** if $p(1, 2) = 1$, $p(L, L - 1) = 1$. 
1. Stationary distributions

**Definition 1.5**

Let \((X_n : n \in \mathbb{N}_0)\) be a homogeneous DTMC with state space \(S\). The distribution \(\pi(x)\), \(x \in S\) is called **stationary** if for all \(y \in S\)

\[
\sum_{x \in S} \pi(x)p(x, y) = \pi(y) \quad \text{or} \quad \langle \pi | P = \langle \pi | .
\]

\(\pi\) is called **reversible** if it fulfills the **detailed balance** conditions

\[
\pi(x)p(x, y) = \pi(y)p(y, x) \quad \text{for all } x, y \in S .
\]

- Reversibility implies stationarity, since
  \[
  \sum_{x \in S} \pi(x)p(x, y) = \sum_{x \in S} \pi(y)p(y, x) = \pi(y) .
  \]

- Stationary distributions as row vectors \(\langle \pi | = (\pi(x) : x \in S)\)
  are **left eigenvectors** with **eigenvalue** \(1\): \(\langle \pi | = \langle \pi | P .\)
1. Absorbing states

**Definition 1.6**

A state \( s \in S \) is called **absorbing** for a DTMC with transition matrix \( p(x, y) \), if

\[
p(s, y) = \delta_{s,y} \quad \text{for all } y \in S.\]

**RW with absorbing BC.**

Let \( h_k \) be the **absorption probability** for \( X_0 = k \in S = \{1, \ldots, L\} \),

\[
h_k = \mathbb{P}[\text{absorption}|X_0 = k] = \mathbb{P}[X_n \in \{1, L\} \text{ for some } n \geq 0|X_0 = k].
\]

Conditioning on the first jump and using Markov, we have the recursion

\[
h_k = ph_{k+1} + qh_{k-1} \quad \text{for } k = 2, \ldots, L - 1; \quad h_1 = h_L = 1.
\]

**Ansatz for solution** \( h_k = \lambda^k, \ \lambda \in \mathbb{C} \):

\[
\lambda = p\lambda^2 + q \quad \Rightarrow \quad \lambda_1 = 1, \quad \lambda_2 = q/p
\]

**General solution** of 2nd order linear recursion

\[
h_k = a\lambda_1^k + b\lambda_2^k = a + b(q/p)^k, \quad a, b \in \mathbb{R}.
\]

Determine coefficients from boundary condition \( \Rightarrow \quad h_k \equiv 1 \)
1. Distribution at time $n$
Consider a DTMC on a finite state space with $|S| = L$, and let $\lambda_1, \ldots, \lambda_L \in \mathbb{C}$ be the eigenvalues of the transition matrix $P$ with corresponding

left (row) eigenvectors $\langle u_i |$ and right (column) eigenvectors $| v_i \rangle$

in bra-ket notation. Assuming that all eigenvalues are distinct we have

$$P = \sum_{i=1}^{L} \lambda_i | v_i \rangle \langle u_i |$$

and

$$P^n = \sum_{i=1}^{L} \lambda_i^n | v_i \rangle \langle u_i |$$

since eigenvectors can be chosen orthonormal $\langle u_i | v_j \rangle = \delta_{i,j}$.

Since $\langle \pi_n | = \langle \pi_0 | P^n$ we get

$$\langle \pi_n | = \langle \pi_0 | v_1 \rangle \lambda_1^n \langle u_1 | + \ldots + \langle \pi_0 | v_L \rangle \lambda_L^n \langle u_L | .$$

- The Gershgorin theorem implies that $|\lambda_i| \leq 1$ and contributions with $|\lambda_i| < 1$ decay exponentially (see hand-out 1).
- $\lambda_1 = 1$ corresponds to the stationary distribution $\langle \pi | = \langle u_1 |$ and $| v_1 \rangle = |1\rangle$.
- Other $\mathbb{C} \ni \lambda_i \neq 1$ with $|\lambda_i| = 1$ correspond to persistent oscillations.
1. Lazy Markov chains

**Definition 1.7**

Let \((X_n : n \in \mathbb{N}_0)\) be a DTMC with transition matrix \(p(x, y)\). The DTMC with transition matrix

\[
p^\epsilon(x, y) = \epsilon \delta_{x,y} + (1 - \epsilon) p(x, y), \quad \epsilon \in (0, 1)
\]

is called a **lazy version** of the original chain.

- **\(P^\epsilon\)** has the **same eigenvectors** as \(P\) with eigenvalues \(\lambda_i^\epsilon = \lambda_i(1-\epsilon) + \epsilon\) since

  \[
  \langle u_i | P^\epsilon = \epsilon \langle u_i | + \lambda_i (1-\epsilon) \langle u_i |
  \]

  (analogously for \(|v_i\rangle\))

- This implies \(|\lambda_i^\epsilon| < |\lambda_i| \leq 1\) unless \(\lambda_i = 1\).
  Such a matrix \(P^\epsilon\) is called **aperiodic**, and there are no persistent oscillations.

- The stationary distribution is unique if and only if the eigenvalue \(\lambda = 1\) is unique (has multiplicity 1), which is independent of lazyness (discussed later).
2. Continuous-time Markov chains

Definition 2.1

A **continuous-time stochastic process** with state space $S$ is a family $(X_t : t \geq 0)$ of random variables taking values in $S$. The process is called **Markov**, if for all $A \subseteq S$, $n \in \mathbb{N}$, $t_1 < \ldots < t_{n+1} \in [0, \infty)$ and $s_1, \ldots, s_n \in S$

$$
P(X_{t_{n+1}} \in A | X_{t_n} = s_n, \ldots, X_{t_1} = s_1) = P(X_{t_{n+1}} \in A | X_{t_n} = s_n) .$$

A Markov process (MP) is called **homogeneous** if for all $A \subseteq S$, $t, u > 0$ and $s \in S$

$$
P(X_{t+u} \in A | X_u = s) = P(X_t \in A | X_0 = s) .$$

If $S$ is discrete, the MP is called a continuous-time **Markov chain** (CTMC).

The generic probability space $\Omega$ of a CTMC is the space of **right-continuous paths**

$$
\Omega = D([0, \infty), S) := \{ X : [0, \infty) \to S \mid X_t = \lim_{u \searrow t} X_u \} 
$$

$P$ is a probability distribution on $\Omega$, which by **Kolmogorov’s extension theorem** is fully specified by its **finite dimensional distributions** (FDDs) of the form

$$
P[X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n], \quad n \in \mathbb{N}, \ t_i \in [0, \infty), \ A_i \subseteq S .$$
2. Continuous-time Markov chains

Proposition 2.1

Let \((X_t : t \geq 0)\) be a homogeneous CTMC with state space \(S\). Then for all \(t \geq 0\) the transition function

\[
p_t(x, y) := P[X_t = y | X_0 = x] = P[X_{t+u} = y | X_u = x] \quad \text{for all } u \geq 0
\]

is well defined and fulfills the Chapman Kolmogorov equations

\[
p_{t+u}(x, y) = \sum_{z \in S} p_t(x, z) p_u(z, y) \quad \text{for all } t, u \geq 0, x, y \in S.
\]

In matrix notation \(P_t = (p_t(x, y) : x, y \in S)\) we get

\[
P_{t+u} = P_t P_u \quad \text{with} \quad P_0 = I.
\]

In particular

\[
\frac{P_{t+\Delta t} - P_t}{\Delta t} = P_t \frac{P_{\Delta t} - I}{\Delta t} = \frac{P_{\Delta t} - I}{\Delta t} P_t,
\]

taking \(\Delta t \searrow 0\) we get the so-called forward and backward equations

\[
\frac{d}{dt} P_t = P_t G = GP_t, \quad \text{where} \quad G = \left. \frac{dP_t}{dt} \right|_{t=0}
\]

is called the generator of the process (sometimes also \(Q\)-matrix).
2. Continuous-time Markov chains

- The solution is given by the matrix exponential

\[ P_t = \exp(tG) = \sum_{k=0}^{\infty} \frac{t^k}{k!} G^k = \mathbb{I} + tG + \frac{t^2}{2}G^2 + \ldots \]  

(2.1)

- The distribution \( \pi_t \) at time \( t > 0 \) is then given by

\[ \langle \pi_t \rangle = \langle \pi_0 \rangle \exp(tG) \] which solves \[ \frac{d}{dt} \langle \pi_t \rangle = \langle \pi_t \rangle G. \]  

(2.2)

- On a finite state space with \( \lambda_1, \ldots, \lambda_L \in \mathbb{C} \) eigenvalues of \( G \), \( P_t \) has eigenvalues \( \exp(t\lambda_i) \) with the same eigenvectors \( \langle v_i |, \ | u_i \rangle \).

If the \( \lambda_i \) are distinct, we can expand the initial condition in the eigenvector basis

\[ \langle \pi_0 \rangle = \alpha_1 \langle v_1 \rangle + \ldots + \alpha_L \langle v_L \rangle, \]

where \( \alpha_i = \langle \pi_0 | u_i \rangle \). This leads to

\[ \langle \pi_t \rangle = \alpha_1 \langle v_1 \rangle e^{\lambda_1 t} + \ldots + \alpha_L \langle v_L \rangle e^{\lambda_L t}. \]  

(2.3)
2. Continuous-time Markov chains

- using (2.1) we have for \( G = (g(x, y) : x, y \in S) \)

\[
p_{\Delta t}(x, y) = g(x, y) \Delta t + o(\Delta t) \quad \text{for all } x \neq y \in S.
\]

So \( g(x, y) \geq 0 \) can be interpreted as transition rates.

\[
p_{\Delta t}(x, x) = 1 + g(x, x) \Delta t + o(\Delta t) \quad \text{for all } x \in S,
\]

and since \( \sum_y p_{\Delta t}(x, y) = 1 \) this implies that

\[
g(x, x) = - \sum_{y \neq x} g(x, y) \leq 0 \quad \text{for all } x \in S.
\]

- (2.2) can then be written intuitively as the Master equation

\[
\frac{d}{dt}\pi_t(x) = \sum_{y \neq x} \pi_t(y) g(y, x) - \sum_{y \neq x} \pi_t(x) g(x, y) \quad \text{for all } x \in S.
\]

\[\begin{array}{c}
\text{gain term} \\
\text{loss term}
\end{array}\]

- The Gershgorin theorem now implies that either \( \lambda_i = 0 \) or \( \text{Re}(\lambda_i) < 0 \) for the eigenvalues of \( G \), so there are no persistent oscillations for CTMCs.
2. Stationary distributions

**Definition 2.2**

Let \((X_t : t \geq 0)\) be a homogeneous CTMC with state space \(S\). The distribution \(\pi(x), x \in S\) is called **stationary** if \(\langle \pi | G = \langle 0 |\), or for all \(y \in S\)

\[
\sum_{x \in S} \pi(x) g(x, y) = \sum_{x \neq y} (\pi(x) g(x, y) - \pi(y) g(y, x)) = 0 .
\] (2.4)

\(\pi\) is called **reversible** if it fulfills the **detailed balance conditions**

\[
\pi(x) g(x, y) = \pi(y) g(y, x) \quad \text{for all } x, y \in S .
\] (2.5)

- again, **reversibility implies stationarity**, since with (2.5) every single term in the sum (2.4) vanishes

- Stationary distributions are left **eigenvectors** of \(G\) with **eigenvalue** 0.

- \(\langle \pi | G = \langle 0 |\) implies \(\langle \pi | P_t = \langle \pi | (I + \sum_{k \geq 1} t^k G^k / k!) = \langle \pi | \) for all \(t \geq 0\)
2. Stationary distributions

**Proposition 2.2 (Existence)**

A DTMC or CTMC with **finite** state space $S$ has **at least one** stationary distribution.

**Proof.** Since $P$ and $G$ have row sum 1 and 0 we have $P\left|\mathbf{1}\right\rangle = \left|\mathbf{1}\right\rangle$ and $G\left|\mathbf{1}\right\rangle = \left|\mathbf{0}\right\rangle$

So 1 and 0 are eigenvalues, and left eigenvectors can be shown to have non-negative entries and thus can be normalized to be stationary distributions $\left\langle \pi \right|$.

**Remark.** If $S$ is countably infinite, stationary distributions may not exist, as for example for the SRW on $\mathbb{Z}$ or the Poisson process on $\mathbb{N}$ (see later).

**Definition 2.3**

A CTMC (or DTMC) is called **irreducible**, if for all $x, y \in S$

$$p_t(x, y) > 0 \text{ for some } t > 0 \quad (p_n(x, y) > 0 \text{ for some } n \in \mathbb{N}) .$$

**Remark.** For continuous time irreducibility implies $p_t(x, y) > 0$ for **all** $t > 0$. 
2. Stationary distributions

**Proposition 2.3 (Uniqueness)**

An irreducible Markov chain has at most one stationary distribution.

**Proof.** Follows from the Perron Frobenius theorem:
Let $P$ be a stochastic matrix ($P = P_t$ for any $t \geq 0$ for CTMCs). Then

1. $\lambda_1 = 1$ is an eigenvalue of $P$, it is singular if and only if the chain is irreducible. Corresponding left and right eigenvectors have non-negative entries.

2. if the chain is continuous-time or discrete-time aperiodic, all remaining eigenvalues $\lambda_i \in \mathbb{C}$, $i \neq 1$ satisfy $\text{Re}(\lambda_i) < 0$ or $|\lambda_i| < 1$, respectively

The second part of the Perron Frobenius theorem also implies convergence of the transition functions to the stationary distribution, since

$$p_t(x, y) = \sum_{i=1}^{|S|} \langle \delta_x | u_i \rangle \langle v_i | e^{\lambda_i t} \rightarrow \langle v_1 | = \langle \pi \rangle \quad \text{as } t \rightarrow \infty.$$
2. Sample paths

Sample paths \( t \mapsto X_t(\omega) \) are piecewise constant and right-continuous by convention.

For \( X_0 = x \), define the holding time \( W_x := \inf \{ t > 0 : X_t \neq x \} \).

### Proposition 2.4

\( W_x \sim \text{Exp}(|g(x,x)|) \), i.e. it is exponentially distributed with mean \( 1/|g(x,x)| \), and if \( |g(x,x)| > 0 \) the chain jumps to \( y \neq x \) after time \( W_x \) with probability \( g(x,y)/|g(x,x)| \).

**Proof.** \( W_x \) has the memoryless property, i.e. for all \( t, u > 0 \)

\[
P(W_x > t + u | W_x > t) = P(W_x > t + u | X_t = x) = P(W_x > u)
\]

where we used the Markov property and homogeneity. Therefore

\[
P(W_x > t + u) = P(W_x > u)P(W_x > t) \quad \Rightarrow \quad P(W_x > t) = e^{\gamma t}
\]

where \( \gamma = \frac{d}{dt}P(W_x > t) \bigg|_{t=0} = \lim_{\Delta t \searrow 0} \frac{p_{\Delta t}(x,x) + o(\Delta t) - 1}{\Delta t} = g(x,x) \leq 0 \).

Conditioned on leaving the current state shortly, the probability to jump to \( y \) is

\[
\lim_{\Delta t \searrow 0} \frac{p_{\Delta t}(x,y)}{1 - p_{\Delta t}(x,x)} = \lim_{\Delta t \searrow 0} \frac{\Delta t g(x,y)}{1 - 1 - \Delta t g(x,x)} = \frac{g(x,y)}{-g(x,x)}.
\]
2. Sample paths

- the **jump times** $J_0, J_1, \ldots$ are defined recursively as

  \[ J_0 = 0 \quad \text{and} \quad J_{n+1} = \inf\{t > J_n : X_t \neq X_{J_n}\}. \]

- due to right-continuous paths, jump times are **stopping times**, i.e. for all $t \geq 0$, the event $\{J_n \leq t\}$ depends only on $(X_s : 0 \leq s \leq t)$.

- By the **strong Markov property** (allows conditioning on state at stopping time), subsequent holding times and jump probabilities are all independent.

- The **jump chain** $(Y_n : n \in \mathbb{N}_0)$ with $Y_n := X_{J_n}$ is then a discrete-time Markov chain with transition matrix

  \[
  p^Y(x, y) = \begin{cases} 
  0, & x = y \quad \text{if } g(x, x) < 0 \quad \text{and} \\
  g(x, y)/|g(x, x)|, & x \neq y \quad \text{if } g(x, x) < 0 \quad \text{and} \\
  \delta_{x,y} & \text{if } g(x, x) = 0 \quad (by \ convention) \ .
  \end{cases}
  \]

- A **sample path** is constructed by simulating the jump chain $(Y_n : n \in \mathbb{N}_0)$ together with independent **holding times** $(W_{Y_n} : n \in \mathbb{N}_0)$, so that $J_n = \sum_{k=0}^{n-1} W_{Y_k}$.
2. Examples

- **A Poisson process** with rate $\lambda$ (short PP($\lambda$)) is a CTMC with

  $$S = \mathbb{N}_0, \; X_0 = 0 \quad \text{and} \quad g(x, y) = \lambda \delta_{x+1,y} - \lambda \delta_{x,y}.$$ 

  The PP($\lambda$) has **stationary and independent increments** with

  $$\mathbb{P}[X_{t+u} = n + k | X_u = n] = p_t(0, k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \quad \text{for all} \; u, t > 0, \; k, n \in \mathbb{N}_0$$

  since $\pi_t(k) = p_t(0, k)$ solves the Master equation $\frac{d}{dt} \pi_t(k) = (\pi_t G)(k)$.

- **A birth-death chain** with **birth rates** $\alpha_x$ and **death rates** $\beta_x$ is a CTMC with

  $$S = \mathbb{N}_0 \quad \text{and} \quad g(x, y) = \alpha_x \delta_{x+1,y} + \beta_x \delta_{x-1,y} - (\alpha_x + \beta_x) \delta_{x,y},$$

  where $\beta_0 = 0$.

  Special cases include

  - **M/M/1 server queues**: $\alpha_x \equiv \alpha > 0, \; \beta_x \equiv \beta > 0 \; \text{for} \; x > 1$
  - **M/M/$\infty$ server queues**: $\alpha_x \equiv \alpha > 0, \; \beta_x = x\beta$
  - **population growth model**: $\alpha_x = x\alpha, \; \beta_x = x\beta$
2. Ergodicity

**Definition 2.4**

A Markov process is called **ergodic** if it has a unique stationary distribution $\pi$ and

$$p_t(x, y) = \mathbb{P}[X_t = y | X_0 = x] \to \pi(y) \text{ as } t \to \infty, \text{ for all } x, y \in S.$$ 

**Theorem 2.5**

An **irreducible** (aperiodic) MC with finite state space is **ergodic**.

**Theorem 2.6 (Ergodic Theorem)**

Consider an **ergodic Markov chain** with unique stationary distribution $\pi$. Then for every bounded function $f : S \to \mathbb{R}$ we have with probability 1

$$\frac{1}{T} \int_0^T f(X_t) \, dt \quad \text{or} \quad \frac{1}{N} \sum_{n=1}^N f(X_n) \to \mathbb{E}_\pi[f] \text{ as } T, N \to \infty.$$ 

- for a proof see e.g. [GS], chapter 9.5
- in practice, use relaxation/burn-in time before computing time averages
2. Markov Chain Monte Carlo (MCMC)

Typical problems related to sampling from $\pi$ on a very large state space $S$

- Compute expectations $\mathbb{E}_\pi [f] = \sum_{x \in S} f(x) \pi(x)$
- For Gibbs measures $\pi(x) = \frac{1}{Z(\beta)} e^{-\beta H(x)}$ (stoch. mech. problems), compute partition function $Z(\beta) = \sum_{x \in S} e^{-\beta H(x)}$

Use the ergodic theorem to estimate expectations by time averages

- Assume $\pi(x) > 0$ for all $x \in S$ (otherwise restrict $S$)
- Invent CTMC/DTMC such that $\pi$ is stationary, e.g. via detailed balance

$$\pi(x) g(x, y) = \pi(y) g(y, x) \quad \text{or} \quad \pi(x) p(x, y) = \pi(y) p(y, x)$$

For Gibbs measures $e^{-\beta H(x)} g(x, y) = e^{-\beta H(y)} g(y, x)$

Typically $g(x, y) = q(x, y) a(x, y)$, i.e. propose move from $x$ to $y$ with rate $q(x, y) = q(y, x)$ (irreducible on $S$ but ’local’), accept with probability $a(x, y)$

- **Heat bath algorithm:** $a(x, y) = \frac{e^{-\beta H(y)}}{e^{-\beta H(x)} + e^{-\beta H(y)}}$
- **Metropolis-Hastings:** $a(x, y) = \begin{cases} 1, & \text{if } H(y) \leq H(x) \\ e^{\beta (H(x) - H(y))}, & \text{if } H(y) > H(x) \end{cases}$
2. Reversibility

Proposition 2.7 (Time reversal)

Let \((X_t : t \in [0, T])\) be a finite state, irreducible CTMC with generator \(G^X\) on a compact time interval which is stationary, i.e. \(X_t \sim \pi\) for \(t \in [0, T]\). Then the time reversed chain

\[ (Y_t : t \in [0, T]) \quad \text{with} \quad Y_t := X_{T-t} \]

is a stationary CTMC with generator \(g^Y(x, y) = \frac{\pi(y)}{\pi(x)} g^X(y, x)\) and stat. prob. \(\pi\).

- An analogous statement holds for stationary, finite state, irreducible DTMCs with

\[ p^Y(x, y) = \frac{\pi(y)}{\pi(x)} p^X(y, x) \]

- Stationary chains with reversible \(\pi\) are time-reversible, \(g^Y(x, y) = g^X(x, y)\).

- The definition of stationary chains can be extended to negative times, \((X_t : t \in \mathbb{R})\), with the time reversed chain given by \(Y_t := X_{-t}\).

- The time reversal of non-stationary MCs is in general not a homogeneous MC, for DTMCs using Bayes’ Theorem we get

\[ p^Y(x, y; n) = \frac{\pi_{N-n-1}(y)}{\pi_{N-n}(x)} p^X(y, x) \]
2. Countably infinite state space

For infinite state space, Markov chains can get ‘lost at infinity’ and have no stationary distribution. Let \( T_x := \inf\{t > J_1 : X_t = x\} \) be the first return time to a state \( x \).

(For DTMCs return times are defined as \( T_x := \inf\{n \geq 1 : X_n = x\} \))

**Definition 2.5**

A state \( x \in S \) is called

- **transient**, if \( \mathbb{P}[T_x = \infty|X_0 = x] > 0 \)
- **null recurrent**, if \( \mathbb{P}[T_x < \infty|X_0 = x] = 1 \) and \( \mathbb{E}[T_x|X_0 = x] = \infty \)
- **positive recurrent**, if \( \mathbb{P}[T_x < \infty|X_0 = x] = 1 \) and \( \mathbb{E}[T_x|X_0 = x] < \infty \)

and these properties partition \( S \) into **communicating classes**.

- For an irreducible MC all states are either transient, null or positive recurrent.
- A MC has a unique stationary distribution if and only if it is positive recurrent

and in this case \( \pi(x) = \frac{1}{\mathbb{E}[T_x|X_0 = x]} \mathbb{E}\left[ \int_0^{T_x} 1_x(X_s) ds | X_0 = x \right] \).
2. Countably infinite state space
A CTMC with an infinite transient component in $S$ can exhibit explosion.

**Definition 2.6**

For a CTMC define the explosion time

$$ J_\infty := \lim_{n \to \infty} J_n \in (0, \infty] \quad \text{where } J_n \text{ are the jump times of the chain}.$$ 

The chain is called **non-explosive** if $\Pr[J_\infty = \infty] = 1$, otherwise it is **explosive**.

- If the exit rates are uniformly bounded, i.e. $\sup_{x \in S} |g(x, x)| < \infty$, then the chain is non-explosive, which is always the case if $S$ is finite.
- As an example, consider a **pure birth chain** with $X_0 = 1$ and rates

$$ g(x, y) = \alpha_x \delta_{y,x+1} - \alpha_x \delta_{y,x}, \quad x, y \in S = \mathbb{N}_0. $$

If $\alpha_x \to \infty$ fast enough (e.g. $\alpha_x = x^2$) we get

$$ \mathbb{E}[J_\infty] = \sum_{x=1}^\infty \mathbb{E}[W_x] = \sum_{x=1}^\infty \frac{1}{\alpha_x} < \infty $$

since holding times $W_x \sim \text{Exp}(\alpha_x)$. This implies $\Pr[J_\infty = \infty] = 0 < 1$. 
3. Markov processes with $S = \mathbb{R}$

**Proposition 3.1**

Let $(X_t : t \geq 0)$ be a homogeneous MP as in Definition 18 with state space $S = \mathbb{R}$. Then for all $t \geq 0$ and (measurable) $A \subseteq \mathbb{R}$ the **transition kernel** for all $x \in \mathbb{R}$

$$P_t(x, A) := \mathbb{P}[X_t \in A | X_0 = x] = \mathbb{P}[X_{t+u} \in A | X_u = x] \quad \text{for all } u \geq 0$$

is well defined. If it is absolutely continuous the **transition density** $p_t$ with

$$P_t(x, A) = \int_A p_t(x, y) \, dy$$

exists and fulfills the **Chapman Kolmogorov equations**

$$p_{t+u}(x, y) = \int_{\mathbb{R}} p_t(x, z) \, p_u(z, y) \, dz \quad \text{for all } t, u \geq 0, \ x, y \in \mathbb{R}.$$  

As for CTMCs, the transition densities and the initial distribution $p_0(x)$ describe all **finite dimensional distributions (fdds)**

$$\mathbb{P}[X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n] = \int_{\mathbb{R}} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} dz_0 p_0(z_0) \, p_{t_1}(z_0, z_1) \cdots p_{t_n-t_{n-1}}(z_{n-1}, z_n)$$
3. Jump processes

\((X_t : t \geq 0)\) is a **jump process** with state space \(S = \mathbb{R}\) characterized by a **jump rate density** \(r(x, y) \geq 0\) with a uniformly bounded **total exit rate** \(R(x) = \int_{\mathbb{R}} r(x, y) \, dy < \bar{R} < \infty\) for all \(x \in \mathbb{R}\).

**Ansatz** for transition function as \(\Delta t \to 0\):

\[
p_{\Delta t}(z, y) = r(z, y) \Delta t + (1 - R(z) \Delta t) \delta(y - z)
\]

Then use the Chapman Kolmogorov equations

\[
p_{t+\Delta t}(x, y) - p_t(x, y) = \int_{\mathbb{R}} p_t(x, z) p_{\Delta t}(z, y) \, dz - p_t(x, y) =
\]

\[
= \int_{\mathbb{R}} p_t(x, z) r(z, y) \Delta t \, dz + \int_{\mathbb{R}} (1 - R(z) \Delta t - 1) p_t(x, z) \delta(y - z) \, dz
\]

to get the **Kolmogorov-Feller equation** \((x \text{ is a fixed initial condition})\)

\[
\frac{\partial}{\partial t} p_t(x, y) = \int_{\mathbb{R}} \left( p_t(x, z) r(z, y) - p_t(x, y) r(y, z) \right) \, dz .
\]

As for CTMC sample paths \(t \mapsto X_t(\omega)\) are piecewise constant and right-continuous.
3. Gaussian processes

$X = (X_1, \ldots, X_n) \sim \mathcal{N}(\mu, \Sigma)$ is a **multivariate Gaussian** in $\mathbb{R}^n$ if it has PDF

$$f(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp \left( -\frac{1}{2} \langle x - \mu | \Sigma^{-1} | x - \mu \rangle \right),$$

with **mean** $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ and **covariance matrix**

$$\Sigma = (\sigma_{ij} : i, j = 1, \ldots, n), \quad \sigma_{ij} = \text{Cov}[X_i, X_j] = \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)].$$

**Definition 3.1**

A stochastic process $(X_t : t \geq 0)$ with state space $S = \mathbb{R}$ is a **Gaussian process** if for all $n \in \mathbb{N}, 0 \leq t_1 < \ldots < t_n$ the vector $(X_{t_1}, \ldots, X_{t_n})$ is a multivariate Gaussian.

**Proposition 3.2**

All fdds of a Gaussian process $(X_t : t \geq 0)$ are fully characterized by the **mean** and the **covariance function**

$$m(t) := \mathbb{E}[X_t] \quad \text{and} \quad \sigma(s, t) := \text{Cov}[X_s, X_t].$$
3. Stationary independent increments

Definition 3.2

A stochastic process \((X_t : t \geq 0)\) has **stationary increments** if

\[ X_t - X_s \sim X_{t-s} - X_0 \quad \text{for all } 0 \leq s \leq t . \]

It has **independent increments** if for all \(n \geq 1\) and \(0 \leq t_1 < \cdots < t_n\)

\[ \{X_{t_{k+1}} - X_{t_k} : 1 \leq k < n\} \quad \text{are independent} . \]

**Example.** The Poisson process \((N_t : t \geq 0) \sim PP(\lambda)\) has stationary independent increments with \(N_t - N_s \sim \text{Poi}(\lambda(t - s))\).

**Proposition 3.3**

The following two statements are equivalent for a stochastic process \((X_t : t \geq 0)\):

- \(X_t\) has stationary independent increments and \(X_t \sim \mathcal{N}(0, t)\) for all \(t \geq 0\).
- \(X_t\) is a Gaussian process with \(m(t) = 0\) and \(\sigma(s, t) = \min\{s, t\}\).

Stationary independent incr. have **stable distributions** such as Gaussian or Poisson.
3. Brownian motion

**Definition 3.3**

**Standard Brownian motion** \((B_t : t \geq 0)\) is a stochastic process that satisfies either of the two equivalent properties in Proposition 3.3 and has **continuous paths**, i.e.

\[
\mathbb{P}\left[\{\omega : t \mapsto B_t(\omega) \text{ is continuous in } t \geq 0\}\right] = 1.
\]

**Theorem 3.4 (Wiener 1923)**

There exists a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) on which standard Brownian motion exists.

**Proof idea.** Construction on \(\Omega = \mathbb{R}^{[0, \infty)}\), using **Kolmogorov’s extension theorem**: For every ’consistent’ description of finite dimensional distributions (fdds) there exists a ’canonical’ process \(X_t[\omega] = \omega(t)\) characterized by a law \(\mathbb{P}\) on \(\Omega\). The main problem is to show that there exists a ’version’ of the process that has continuous paths, i.e. \(\mathbb{P}\) can be chosen to concentrate on continuous paths \(\omega\).

**Remark.** Construction of \((N_t : t \geq 0) \sim PP(\lambda)\) is

\[
N_t := \max \left\{ k \geq 1 : \tau_1 + \cdots + \tau_k \leq t \right\}, \quad \tau_1, \tau_2, \cdots \sim \text{Exp}(\lambda) \text{ iids}
\]
3. Properties of Brownian motion

- SBM is a time-homogeneous MP with $B_0 = 0$.
- $\sigma B_t + x$ with $\sigma > 0$ is a (general) BM with $B_t \sim \mathcal{N}(x, \sigma^2 t)$.
  The transition density is given by a Gaussian PDF

$$p_t(x, y) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp \left( -\frac{(y-x)^2}{2\sigma^2 t} \right)$$

This is also called the heat kernel, since it solves the heat/diffusion equation

$$\frac{\partial}{\partial t} p_t(x, y) = \frac{\sigma^2}{2} \frac{\partial^2}{\partial y^2} p_t(x, y) \quad \text{with} \quad p_0(x, y) = \delta(y - x).$$

- SBM is self-similar with Hurst exponent $H = 1/2$, i.e.

$$(B_{\lambda t} : t \geq 0) \sim \lambda^H (B_t : t \geq 0) \quad \text{for all } \lambda > 0.$$ 

- $t \mapsto B_t$ is $\mathbb{P} - a.s.$ not differentiable at $t$ for all $t \geq 0$.
  For fixed $h > 0$ define $\xi^h_t := (B_{t+h} - B_t)/h \sim \mathcal{N}(0, 1/h)$, which is a mean-0 Gaussian process with covariance $\sigma(s, t) = \begin{cases} 0, & |t - s| > h \\ (h - |t - s|)/h^2, & |t - s| < h \end{cases}$.

The (non-existent) derivative $\xi_t := \lim_{h \to 0} \xi^h_t$ is called white noise and is formally a mean-0 Gaussian process with covariance $\sigma(s, t) = \delta(t - s)$. 
3. Generators as operators

For a CTMC \((X_t : t \geq 0)\) with discrete state space \(S\) we have for \(f : S \to \mathbb{R}\)

\[
\mathbb{E}[f(X_t)] = \sum_{x \in S} \pi_t(x)f(x) = \langle \pi_t | f \rangle \quad \text{and} \quad \frac{d}{dt} \langle \pi_t | = \langle \pi_t | G
\]

Therefore

\[
\frac{d}{dt} \mathbb{E}[f(X_t)] = \frac{d}{dt} \langle \pi_t | f \rangle = \langle \pi_t | G | f \rangle = \mathbb{E}[(Gf)(X_t)]
\]

The generator \(G\) can be defined as an operator \(G\) acting on functions \(f : S \to \mathbb{R}\)

\[
G|f\rangle(x) = (Gf)(x) = \sum_{y \neq x} g(x, y) [f(y) - f(x)]
\]

For Brownian motion use the heat eq. and integration by parts for \(f \in C^2(\mathbb{R})\)

\[
\frac{d}{dt} \mathbb{E}_x[f(X_t)] = \int_{\mathbb{R}} \partial_t p_t(x, y)f(y)dy = \frac{\sigma^2}{2} \int_{\mathbb{R}} \partial^2_y p_t(x, y)f(y)dy = \mathbb{E}_x[(Lf)(X_t)]
\]

where the generator of BM is \((Lf)(x) = \frac{\sigma^2}{2} \Delta f(x) \quad \text{(or} \frac{\sigma^2}{2} f''(x))\).

For jump processes with \(S = \mathbb{R}\) and rate density \(r(x, y)\) the generator is

\[
(Lf)(x) = \int_{\mathbb{R}} r(x, y)[f(y) - f(x)]dy.
\]
3. Brownian motion as scaling limit

Proposition 3.5

Let \( (X_t : t \geq 0) \) be a jump process on \( \mathbb{R} \) with translation invariant rates \( r(x, y) = q(y - x) \) which have

- mean zero \( \int_{\mathbb{R}} q(z) z \, dz = 0 \) and

- finite second moment \( \sigma^2 := \int_{\mathbb{R}} q(z) z^2 \, dz < \infty \).

Then for all \( T > 0 \) the rescaled process

\[
\left( \epsilon X_{t/\epsilon^2} : t \in [0, T] \right) \Rightarrow \left( B_t : t \in [0, T] \right) \quad \text{as } \epsilon \to 0
\]

converges in distribution to a BM with generator \( \mathcal{L} = \frac{1}{2} \sigma^2 \Delta \) for all \( T > 0 \).

Proof. Taylor expansion of the generator for test functions \( f \in C^3(\mathbb{R}) \), and tightness argument for continuity of paths (requires fixed interval \([0, T] \)).
3. Diffusion processes

**Definition 3.4**

A **diffusion process** with drift \(a(x, t) \in \mathbb{R}\) and diffusion \(\sigma(x, t) > 0\) is a real-valued process with continuous paths and generator

\[
(Lf)(x) = a(x, t)f'(x) + \frac{1}{2} \sigma^2(x, t)f''(x).
\]

**Examples.**

- The **Ornstein-Uhlenbeck process** is a diffusion process with generator

  \[
  (Lf)(x) = -\alpha x f'(x) + \frac{1}{2} \sigma^2 f''(x), \quad \alpha, \sigma^2 > 0.
  \]

  It has a Gaussian stationary distribution \(\mathcal{N}(0, \sigma^2/(2\alpha))\). If the initial distribution \(\pi_0\) is Gaussian, this is a **Gaussian process**.

- The **Brownian bridge** is a Gaussian diffusion with \(X_0 = 0\) and generator

  \[
  (Lf)(x) = -\frac{x}{1 - t} f'(x) + \frac{1}{2} f''(x).
  \]

  Equivalently, it can be characterized as a SBM conditioned on \(B_1 = 0\).
3. Diffusion processes

Time evolution of the mean. Use \( \frac{d}{dt} \mathbb{E}[f(X_t)] = \mathbb{E}[(\mathcal{L}f)(x_t)] \) with \( f(x) = x \)

\[
\frac{d}{dt} \mathbb{E}[X_t] = \mathbb{E}[a(X_t, t)]
\]

Time evolution of the transition density. With \( X_0 = x \) we have for \( p_t(x, y) \)

\[
\int_{\mathbb{R}} \frac{\partial}{\partial t} p_t(x, y) f(y) dy = \frac{d}{dt} \mathbb{E}[f(X_t)] = \int_{\mathbb{R}} p_t(x, y) \mathcal{L}f(y) dy \quad \text{for any } f.
\]

Use integration by parts to get the Fokker-Planck equation

\[
\frac{\partial}{\partial t} p_t(x, y) = -\frac{\partial}{\partial y} (a(y, t)p_t(x, y)) + \frac{1}{2} \frac{\partial^2}{\partial y^2} (\sigma^2(y, t)p_t(x, y)).
\]

Stationary distributions for time-independent \( a(y) \in \mathbb{R} \) and \( \sigma^2(y) > 0 \)

\[
\frac{d}{dy} (a(y)p^*(y)) = \frac{1}{2} \frac{d^2}{dy^2} (\sigma^2(y)p^*(y)) ,
\]

leads to a stationary density (modulo normalization fixing \( p^*(0) \))

\[
p^*(x) = p^*(0) \exp \left( \int_0^x \frac{2a(y) - (\sigma^2)'(y)}{\sigma^2(y)} dy \right).
\]
3. Beyond diffusion

**Definition 3.5**

A Lévy process \((X_t : t \geq 0)\) is a real-valued process with right-continuous paths and stationary, independent increments.

The generator has a part with **constant drift** \(a \in \mathbb{R}\) and **diffusion** \(\sigma^2 \geq 0\)

\[
\mathcal{L}f(x) = af'(x) + \frac{\sigma^2}{2} f''(x) + \int_{\mathbb{R}} \left( f(x + z) - f(x) - zf'(x) \mathbb{1}_{(0,1)}(|z|) \right) q(z) dz,
\]
and a translation invariant **jump part** with density \(q(z)\) (or measure \(\nu(dz)\)) and fulfills

\[
\int_{|z|>1} q(z)dz < \infty \quad \text{and} \quad \int_{0<|z|<1} z^2 q(z)dz < \infty.
\]

- Diffusion processes, in particular **BM** with \(a = 0, \sigma^2 > 0\) and \(q(z) \equiv 0\), or jump processes, in particular **Poisson** with \(a = \sigma = 0\) and \(q(z) = \lambda \delta(z - 1)\).
- For \(a = \sigma = 0\) and heavy-tailed jump distribution

\[
q(z) = \frac{C}{|z|^{1+\alpha}} \quad \text{with} \quad C > 0 \text{ and } \alpha \in (0, 2]
\]

the process is called **\(\alpha\)-stable symmetric Lévy process** or **Lévy flight**.

**self-similar** \((X_{\lambda t} : t \geq 0) \sim \lambda^H (X_t : t \geq 0)\), \(\lambda > 0\) with \(H = 1/\alpha\)

\[
\Rightarrow \quad \text{super-diffusive behaviour} \quad \text{with} \quad \mathbb{E}[X_t^2] \propto t^{2/\alpha}
\]
3. Beyond diffusion

In general, a process \((X_t : t \geq 0)\) is said to exhibit anomalous diffusion if

\[
\text{Var}[X_t]/t \rightarrow \begin{cases} 0, & \text{sub-diffusive} \\ \infty, & \text{super-diffusive} \end{cases} \quad \text{as } t \rightarrow \infty.
\]

**Definition 3.6**

A fractional Brownian motion (fBM) \((B^H_t : t \geq 0)\) with Hurst index \(H \in (0, 1)\) is a mean-zero Gaussian process with continuous paths, \(B^H_0 = 0\) and covariances

\[
\mathbb{E}[B^H_t B^H_s] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right) \quad \text{for all } s, t \geq 0.
\]

- For \(H = 1/2\), fBM is standard Brownian motion.
- fBM has stationary Gaussian increments where for all \(t > s \geq 0\)
  \[
  B^H_t - B^H_s \sim B^H_{t-s} \sim \mathcal{N}(0, (t-s)^{2H}),
  \]
  which for \(H \neq 1/2\) are not independent and the process is non-Markov.
- fBM is self-similar, i.e. \((B^H_{\lambda t} : t \geq 0) \sim \lambda^H (B^H_t : t \geq 0)\) for all \(\lambda > 0\).
3. Fractional BM and noise

- fBM exhibits **anomalous diffusion** with \( \text{Var}[B_t^H] = t^{2H} \)
- \( H > 1/2 \): super-diffusive with positively correlated increments
- \( H < 1/2 \): sub-diffusive with negatively correlated increments

\[
\mathbb{E}[B_1^H(B_{t+1}^H - B_t^H)] = \frac{(t+1)^{2H} - 2t^{2H} + (t-1)^{2H}}{2} \xrightarrow{t \to \infty} H(2H-1)t^{2(H-1)}
\]

For a **stationary process** \((X_t : t \geq 0)\) on \(\mathbb{R}\) define the **autocorrelation/covariance fct**

\[
c(t) := \text{Cov}[X_s, X_{s+t}] \quad \text{for all } s, t \in \mathbb{R}.
\]

Its Fourier transform is the **spectral density** \( S(\omega) := \int_\mathbb{R} c(t)e^{-i\omega t}dt \)

- **white noise** \((\xi_t : t \geq 0)\), stationary GP with mean zero and
  \[
c(t) = \delta(t) \quad \Rightarrow \quad S(\omega) \equiv 1.
\]

- **fractional or 1/f noise** \((\xi_t^H : t \geq 0)\), stationary GP with mean zero and
  \[
c(t) = \frac{2H(2H-1)}{|t|^{2(1-H)}} \quad \Rightarrow \quad S(\omega) \propto |\omega|^{2(1-H)-1} = \frac{1}{\omega^{2H-1}}
\]
3. SDEs and Itô’s formula

Let \((B_t : t \geq 0)\) be a standard BM. Then a diffusion process with drift \(a(x, t)\) and diffusion \(\sigma(x, t)\) solves the **Stochastic differential equation (SDE)**

\[
dX_t = a(X_t, t)dt + \sigma(X_t, t)dB_t.
\]

Here \(dB_t\) is white noise, interpreted in integrated form as

\[
X_t - X_0 = \int_0^t a(X_s, s)ds + \int_0^t \sigma(X_s, s)dB_s.
\]

**Theorem 3.6 (Itô’s formula for diffusions)**

Let \((X_t : t \geq 0)\) be a diffusion with generator \(\mathcal{L}\) and \(f : \mathbb{R} \rightarrow \mathbb{R}\) a smooth. Then

\[
f(X_t) - f(X_0) = \int_0^t (\mathcal{L}f)(X_s)ds + \int_0^t \sigma(X_s, s)f'(X_s)dB_s.
\]

or, equivalently in terms of SDEs

\[
df(X_t) = a(X_t, t)f'(X_t)dt + \frac{1}{2}\sigma^2(X_t, t)f''(X_t)dt + \sigma(X_t, t)f'(X_t)dB_t.
\]
3. SDEs and Itô’s formula

Itô’s formula for diffusions implies the following.

**Proposition 3.7**

Let \((X_t : t \geq 0)\) be a diffusion process with drift \(a(x, t)\) and diffusion \(\sigma(x, t)\), and \(f : \mathbb{R} \rightarrow \mathbb{R}\) a smooth invertible function. Then \((Y_t : t \geq 0)\) with \(Y_t = f(X_t)\) is a diffusion process with \((x = f^{-1}(y))\)

\[
\text{drift} \quad a(x, t)f'(x) + \frac{1}{2} \sigma^2(x, t)f''(x) \quad \text{and diffusion} \quad \sigma(x, t)f'(x).
\]

**Geometric BM.** \(Y_t := e^{\theta B_t}\), so \(f(x) = e^{\theta x}\) with \(f'(x) = \theta f(x)\) and \(f''(x) = \theta^2 f(x)\), where \((B_t : t \geq 0)\) is standard BM with \(a \equiv 0, \sigma^2 \equiv 1\) and \(\theta \in \mathbb{R}\).

Then \((Y_t : t \geq 0)\) is a diffusion process with SDE \(dY_t = \frac{\theta}{2} Y_t dt + \theta Y_t dB_t\).

**Exponential martingale.**

\(Z_t := e^{\theta B_t - \frac{\theta^2 t}{2}}\), so \(f(x, t) = e^{\theta x - \frac{\theta^2 t}{2}}\) and \(\partial_t f(x, t) = -\frac{\theta^2}{2} f(x, t)\)

Then \(dZ_t = \frac{\theta}{2} Z_t dt - \frac{\theta}{2} Z_t dt + \theta Z_t dB_t = \theta Z_t dB_t\)

and \((Z_t : t \geq 0)\) is a **martingale** (see next slide) with \(\mathbb{E}[Z_t] \equiv Z_0 = 1\).
3. Fluctuations and martingales

**Definition 3.7**

A real-valued stochastic process \((M_t : t \geq 0)\) is a **martingale** w.r.t. the process \((X_t : t \geq 0)\) if for all \(t \geq 0\) we have \(\mathbb{E}[|M_t|] < \infty\) and

\[
\mathbb{E}[M_t \mid \{X_u : 0 \leq u \leq s\}] = M_s \text{ a.s. for all } s \leq t .
\]

If in addition \(\mathbb{E}[M_t^2] < \infty\), there exists a unique increasing process \(([M]_t : t \geq 0)\) called the **quadratic variation**, with \([M]_0 = 0\) and such that \(M_t^2 - [M]_t\) is martingale.

**Theorem 3.8 (Itô’s formula)**

Let \((X_t : t \geq 0)\) be a Markov process on state space \(S\) with generator \(\mathcal{L}\). Then for any smooth enough \(f : S \times [0, \infty) \rightarrow \mathbb{R}\)

\[
f(X_t, t) - f(X_0, 0) = \int_0^t (\mathcal{L}f)(X_s, s) ds + \int_0^t \partial_s f(X_s, s) ds + M^f_t ,
\]

where \((M^f_t : t \geq 0)\) is a martingale w.r.t. \((X_t : t \geq 0)\) with \(M^f_0 = 0\) and the quadratic variation

\[
[M^f]_t = \int_0^t ((\mathcal{L}f^2)(X_s, s) - 2(f \mathcal{L}f)(X_s, s)) ds .
\]
3. Fluctuations and martingales

- For a **Poisson process** \((N_t : t \geq 0)\) with rate \(\lambda > 0\) Itô’s formula implies that
  \[
  M_t := N_t - \lambda t \quad \text{is a martingale with quadr. variation} \quad [M]_t = \lambda t.
  \]

- **Watanabe’s characterisation of PP**: Let \((N_t : t \geq 0)\) be a **counting process**, i.e. a jump process on \(S = \mathbb{N}\) with jump size \(+1\) only. If \(M_t = N_t - \lambda t\) is a martingale, then \((N_t : t \geq 0) \sim PP(\lambda)\).

- For a **diffusion process**, choosing \(f(X_t, t) = X_t\) in Itô’s formula leads to
  \[
  X_t - X_0 = \int_0^t a(X_s, s)ds + M_t \quad \text{with} \quad [M]_t = \int_0^t \sigma^2(X_s, s)ds.
  \]

  In particular for BM with \(a(x, t) \equiv 0\) and \(\sigma^2(x, t) \equiv \sigma^2\) we have
  \[
  (B_t : t \geq 0) \quad \text{is a martingale with quadratic variation} \quad [B]_t = t.
  \]

- **Lévy’s characterisation of BM**: Any continuous martingale \((M_t : t \geq 0)\) on \(\mathbb{R}\) with \(M_0 = 0\) and quadratic variation \([M]_t = t\) is standard Brownian motion.

  Furthermore, **any continuous martingale** \((M_t : t \geq 0)\) on \(\mathbb{R}\) with \(M_0 = 0\) is a continuous (random) time-change of a standard BM, i.e.
  \[
  (M_t : t \geq 0) \sim (B_{[M]} : t \geq 0) \quad \text{for SBM} \quad (B_t : t \geq 0).
  \]
3. Fluctuations and martingales

For a **diffusion process** \((X_t : t \geq 0)\) we have

\[
X_t - X_0 = \int_0^t a(X_s, s) ds + M_t \quad \text{with} \quad [M]_t = \int_0^t \sigma^2(X_s, s) ds.
\]

with \(M_t\) a continuous martingale \(\Rightarrow (M_t : t \geq 0) \sim (B_{[M]} : t \geq 0)\)

- Related time-changed BMs can be written as **stochastic Itô integrals**

\[
M_t = \int_0^t \sigma(X_s, s) dB_s := B_{[M]}.
\]

Therefore \(\sigma \equiv 0\) implies **deterministic dynamics** with \(M_t \equiv 0\),
and the corresponding SDE is an ODE \(dX_t/dt = a(X_t, t)\).

Vanishing drift \(a \equiv 0\) implies \(X_t - X_0 = M_t\) or \(dX_t = \sigma(X_t, t) dB_t\)
and the process \((X_t : t \geq 0)\) is a **martingale**.

- Recall the **exponential martingale** \(e^{\theta B_t - \theta^2 t/2}\) as a non-trivial example.
3. Martingales and conservation laws

Consider a CTMP \((X_t : t \geq 0)\) on state space \(S\) with generator \(\mathcal{L}\), and an observable \(f : S \to \mathbb{R}\) such that \(\mathcal{L}f : S \to \mathbb{R}\) is well defined (e.g. \(f \in C^2(S, \mathbb{R})\) for diffusions).

**Proposition 3.9**

If \(\mathcal{L}f(x) = 0\) for all \(x \in S\), then \(f(X_t)\) is a martingale, and is conserved in expectation, i.e. (for any initial condition \(X_0\))

\[
E[f(X_t)] = E[f(X_0)] \quad \text{for all } t \geq 0 .
\]

If in addition \(\mathcal{L}f^2(x) = 0\) for all \(x \in S\), then \(f(X_t)\) is conserved (or a conserved quantity), i.e. (for any initial condition \(X_0\))

\[
f(X_t) = f(X_0) \quad \text{almost surely for all } t \geq 0 .
\]

**Proof.** The first claim follows directly from Itô’s formula (Theorem 3.8). For the second claim, we have \(f(X_t) = f(X_0) + M^f_t\) and \(M^f_t\) has quadratic variation

\[
[M^f]_t = \int_0^t ((\mathcal{L}f^2)(X_s, s) - 2(f \mathcal{L}f)(X_s, s)) ds = 0 ,
\]

for all \(t \geq 0\), which implies \(M^f_t \equiv 0\) almost surely. \(\square\)
4. Stochastic particle systems

- **lattice/population**: $\Lambda = \{1, \ldots, L\}$, finite set of points
- **state space** $S$ is given by the set of all **configurations**

$$\eta = (\eta(i) : i \in \Lambda) \in S = \{0, 1\}^L$$ (often also written $\{0, 1\}^\Lambda$).

$\eta(i) \in \{0, 1\}$ signifies the presence of a particle/infection at site/individual $i$.
- Only local transitions are allowed with rates

$$\eta \rightarrow \eta^i \quad \text{with rate} \quad c(\eta, \eta^i) \quad \text{(reaction)}$$

$$\eta \rightarrow \eta^{ij} \quad \text{with rate} \quad c(\eta, \eta^{ij}) \quad \text{(transport)}$$

where $\eta^i(k) = \begin{cases} \eta(k) , & k \neq i \\ 1 - \eta(k) , & k = i \end{cases}$ and $\eta^{ij}(k) = \begin{cases} \eta(k) , & k \neq i,j \\ \eta(j) , & k = i \\ \eta(i) , & k = j \end{cases}$

**Definition 4.1**

A **stochastic particle system** is a CTMC with state space $S = \{0, 1\}^\Lambda$ and generator

$$\mathcal{L}f(\eta) = \sum_{i \in \Lambda} c(\eta, \eta^i) [f(\eta^i) - f(\eta)] \quad \text{or} \quad \mathcal{L}f(\eta) = \sum_{i,j \in \Lambda} c(\eta, \eta^{ij}) [f(\eta^{ij}) - f(\eta)].$$
4. Contact process

The contact process is a simple stochastic model for the SI epidemic with infection rates \( q(i,j) \geq 0 \) and uniform recovery rate 1.

**Definition 4.2**

The contact process (CP) \((\eta_t : t \geq 0)\) is an IPS with rates

\[
c(\eta, \eta^i) = \underbrace{1 \cdot \delta_{\eta(i), 1}}_{\text{recovery}} + \underbrace{\delta_{\eta(i), 0} \sum_{j \neq i} q(j, i) \delta_{\eta(j), 1}}_{\text{infection}} \quad \text{for all } i \in \Lambda.
\]

Usually, \( q(i, j) = q(j, i) \in \{0, \lambda\} \), i.e. connected individuals infect each other with fixed rate \( \lambda > 0 \).

- The CP has one absorbing state \( \eta(i) = 0 \) for all \( i \in \Lambda \), which can be reached from every initial configuration. Therefore the process is ergodic and the infection eventually gets **extinct** with probability 1.
- Let \( T := \inf\{t > 0 : \eta_t \equiv 0\} \) be the extinction time. Then there exists a **critical value (epidemic threshold)** \( \lambda_c > 0 \) such that (for irreducible \( q(i, j) \))

  \[
  \mathbb{E}[T|\eta_0 \equiv 1] \propto \log L \quad \text{for } \lambda < \lambda_c \quad \text{and} \quad \mathbb{E}[T|\eta_0 \equiv 1] \propto e^{CL} \quad \text{for } \lambda > \lambda_c.
  \]
4. Voter model

The voter model describes opinion dynamics with influence rates \( q(i, j) \geq 0 \) at which individual \( i \) persuades \( j \) to switch to her/his opinion.

**Definition 4.3**

The linear voter model (VM) \((\eta_t : t \geq 0)\) is an IPS with rates

\[
c(\eta, \eta^i) = \sum_{j \neq i} q(j, i) \left( \delta_{\eta(i), 1}\delta_{\eta(j), 0} + \delta_{\eta(i), 0}\delta_{\eta(j), 1} \right)
\]

for all \( i \in \Lambda \).

In non-linear versions the rates can be replaced by general (symmetric) functions.

- The VM is symmetric under relabelling opinions 0 ↔ 1.
- If \( q(i, j) \) is irreducible there are two absorbing states, \( \eta \equiv 0, 1 \), both of which can be reached from every initial condition. Therefore the VM is not ergodic, and stationary measures are

  \[
  \alpha \delta_0 + (1 - \alpha) \delta_1 \quad \text{with} \quad \alpha \in [0, 1] \quad \text{depending on the initial condition}.
  \]

- Coexistence of both opinions can occur on infinite lattices (e.g. \( \mathbb{Z}^d \) for \( d \geq 3 \)).
4. Exclusion process

The exclusion process describes transport of a conserved quantity (e.g. mass or energy) with **transport rates** \( q(i,j) \geq 0 \) site \( i \) to \( j \).

**Definition 4.4**

The exclusion process (EP) \( (\eta_t : t \geq 0) \) is an IPS with rates

\[
c(\eta, \eta^{ij}) = q(i,j) \delta_{\eta(i),1} \delta_{\eta(j),0} \quad \text{for all } i,j \in \Lambda .
\]

The EP is called **simple (SEP)** if jumps occur only between nearest neighbours on \( \Lambda \). The SEP is **symmetric (SSEP)** if \( q(i,j) = q(j,i) \), otherwise **asymmetric (ASEP)**.

- The SEP is mostly studied in a 1D geometry with periodic or open boundaries.
- For periodic boundary conditions the total number of particles \( N = \sum_i \eta(i) \) is **conserved**. The process is ergodic on the sub-state space

\[
S_N = \left\{ \eta \in \{0,1\}^L : \sum_i \eta(i) = N \right\}
\]

for each value \( N = 0, \ldots, L \), and has a unique stationary distribution.
- For open boundaries particles can be created and destroyed at the boundary, the system is ergodic on \( S \) and has a unique stationary distribution.
5. Graphs - definition

Definition 5.1

A graph (or network) $G = (V, E)$ consists of a finite set $V = \{1, \ldots, N\}$ of vertices (or nodes, points), and a set $E \subseteq V \times V$ of edges (or links, lines). The graph is called undirected if $(i, j) \in E$ implies $(j, i) \in E$, otherwise directed. The structure of the graph is encoded in the adjacency (or connectivity) matrix

$$A = (a_{ij} : i, j \in V) \quad \text{where} \quad a_{ij} = \begin{cases} 1, & (i, j) \in E \\ 0, & (i, j) \notin E \end{cases}.$$

We denote the number of edges by $K = |E|$ for directed, or $K = |E|/2$ for undirected graphs.

- Graphs we consider do not have self edges, i.e. $(i, i) \notin E$ for all $i \in V$, or multiple edges, since edges $(i, j)$ are unique elements of $E$.

- Weighted graphs with edge weights $w_{ij} \in \mathbb{R}$ can be used to represent continuous- or discrete-time Markov chains.

- In general graphs can also be infinite, but we will focus on finite graphs. Many of the following graph characteristics only make sense in the finite case.
5. Graphs - paths and connectivity

**Definition 5.2**

A **path** $\gamma_{ij}$ of length $l = |\gamma_{ij}|$ from vertex $i$ to $j$ is sequence of vertices

$$\gamma_{ij} = (v_1 = i, v_2, \ldots, v_{l+1} = j) \quad \text{with} \quad (v_k, v_{k+1}) \in E \text{ for all } k = 1, \ldots, l,$$

and $v_k \neq v_{k'}$ for all $k \neq k' \in \{1, \ldots, l\}$ (i.e. each vertex is visited only once).

If such a path exists, we say that vertex $i$ is **connected** to $j$ (write $i \rightarrow j$). **Shortest paths** between vertices $i, j$ are called **geodesics** (not necessarily unique) and their length $d_{ij}$ is called the **distance** from $i$ to $j$. If $i \not\leftrightarrow j$ we set $d_{ij} = \infty$.

A graph is **connected** if $d_{ij} < \infty$ for all $i, j \in V$. The **diameter** and the **characteristic path length** of the graph $G$ are given by

$$\text{diam}(G) := \max \{d_{ij} : i, j \in V\} \in \mathbb{N}_0 \cup \{\infty\},$$

$$L = L(G) := \frac{1}{N(N-1)} \sum_{i,j \in V} d_{ij} \in [0, \infty].$$

For undirected graphs we have $d_{ij} = d_{ji}$ which is finite if $i \leftrightarrow j$, and they can be decomposed into **connected components**, where we write

$$C_i = \{j \in V : j \leftrightarrow i\} \quad \text{for the component containing vertex } i.$$
5. Graphs - degrees

**Definition 5.3**

The **in-** and **out-degree** of a node \( i \in V \) is defined as

\[
    k_{i}^{\text{in}} = \sum_{j \in V} a_{ji} \quad \text{and} \quad k_{i}^{\text{out}} = \sum_{j \in V} a_{ij}.
\]

\( k_{1}^{\text{in}}, \ldots k_{N}^{\text{in}} \) is called the **in-degree sequence** and the **in-degree distribution** is

\[
    (p^{\text{in}}(k) : k \in \{0, \ldots, K\}) \quad \text{with} \quad p^{\text{in}}(k) = \frac{1}{N} \sum_{i \in V} \delta_{k, k_{i}^{\text{in}}}
\]

giving the fraction of vertices with in-degree \( k \). The same holds for out-degrees, and in undirected networks we simply write \( k_{i} = k_{i}^{\text{in}} = k_{i}^{\text{out}} \) and \( p(k) \).

- Note that \( \sum_{i \in V} k_{i} = \sum_{i,j \in V} a_{ij} = |E| \) (also for directed), average and variance are

\[
    \langle k \rangle = \frac{1}{N} \sum_{i \in V} k_{i} = |E|/N = \sum_{k} kp(k) , \quad \sigma^{2} = \langle k^{2} \rangle - \langle k \rangle^{2} .
\]

- In a **regular graph** (usually undirected) all vertices have equal degree \( k_{i} \equiv k \).

- Graphs where the degree distribution shows a power law decay, i.e. \( p(k) \propto k^{-\alpha} \) for large \( k \), are often called **scale-free**.

Real-world networks are often scale-free with exponent around \( \alpha \approx 3 \).
5. Graphs - first examples

Example 2 (Some graphs)

The **complete graph** $K_N$ with $N$ vertices is an undirected graph where all $N(N - 1)/2$ vertices $E = \{(i, j) : i \neq j \in V\}$ are present.

**Regular lattices** $\mathbb{Z}^d$ with edges between nearest neighbours are examples of regular graphs with degree $k = 2d$.

Definition 5.4

A **tree** is an undirected graph where any two vertices are connected by exactly one path. Vertices with degree 1 are called **leaves**.

In a **rooted tree** one vertex $i \in V$ is the designated **root**, and the graph can be directed, where all vertices point towards or away from the root.

A **cycle** is a closed path $\gamma_{ii}$ of length $|\gamma_{ii}| > 2$. $G$ is a tree if and only if

- it is connected and has no cycles;
- it is connected but is not connected if a single edge is removed;
- it has no cycles but a cycle is formed if any edge is added.
5. Graphs - degree correlations

Definition 5.5

For undirected graphs, the **joint degree distribution** of nodes linked by an edge is

\[ q(k, k') = \frac{1}{|E|} \sum_{(i,j) \in E} \delta_{k_i,k} \delta_{k_j,k'} = \frac{\sum_{i,j \in V} a_{ij} \delta_{k_i,k} \delta_{k_j,k'}}{\sum_{i,j \in V} a_{ij}} = q(k', k). \]

With the marginal \( q(k') = \sum_k q(k, k') \) we have the **conditional degree distribution**

\[ q(k|k') = q(k, k')/q(k') \quad \text{with average} \quad k_{nn}(k') := \sum_k kq(k|k'). \]

The network is called **uncorrelated** if \( k_{nn}(k') \) is independent of \( k' \), **assortative** if \( k_{nn}(k') \uparrow \) in \( k' \) and **disassortative** if \( k_{nn}(k') \downarrow \) in \( k' \).

- The marginal \( q(k) \) corresponds to **edge biased degree sampling**, i.e.

  \[ q(k) = \sum_{k'} q(k, k') = \frac{1}{|E|} \sum_{i,j \in V} a_{ij} \delta_{k_i,k} = \frac{N}{|E|} \frac{1}{N} \sum_{i \in V} k_i \delta_{k_i,k} = \frac{kp(k)}{\langle k \rangle}. \]

  For uncorrelated networks \( q(k|k') = q(k) \) and thus \( k_{nn}(k') = \langle k^2 \rangle / \langle k \rangle \).
5. Subgraphs

- The degree of correlation can be quantified by the **correlation coefficient**

\[
\chi := \frac{\langle kk' \rangle_q - \langle k \rangle_q^2}{\langle k^2 \rangle_q - \langle k \rangle_q^2} = \frac{\sum_{k,k'} kk' (q(k,k') - q(k)q(k'))}{\sum_k k^2 q(k) - (\sum_k k q(k))^2} \in [-1, 1].
\]

**Definition 5.6**

A **subgraph** \( G' = (V', E') \) of \( G = (V, E) \) is a graph such that \( V' \subseteq V \) and \( E' \subseteq E \).

- Small connected subgraphs are also called **motifs**, the simplest non-trivial examples in undirected graphs are connected triples and triangles.
- Fully connected (complete) subgraphs which are maximal with respect to connectedness are called **cliques**.
- A **spanning tree** is a tree subgraph that contains all vertices of the graph.
- A subgraph \( G' \) is called a **community**, if (for example)

\[
\sum_{i,j \in V'} a_{ij} > \sum_{i \in V', j \notin V'} a_{ij} \quad \text{(there are also other definitions)}.
\]
5. Clustering

Clustering aims to quantify the probability that two neighbours of a given vertex are themselves neighbours. Two different definitions are used in the literature.

**Definition 5.7**

The **global clustering coefficient** for an undirected graph is defined as

\[
C = \frac{3 \times \text{# of (connected) triangles}}{\text{# of (connected) triples}} = \frac{3 \sum_{i<j<l} a_{ij}a_{jl}a_{li}}{\sum_{i<j<l}(a_{ij}a_{il} + a_{ji}a_{jl} + a_{li}a_{lj})} \in [0, 1].
\]

Alternatively, one can define a **local clustering coefficient**

\[
C_i = \frac{\text{# of triangles containing vertex } i}{\text{# of triples centered on vertex } i} = \frac{\sum_{j<l} a_{ij}a_{jl}a_{li}}{\sum_{j<l} a_{ij}a_{il}} \in [0, 1],
\]

and use the average \( \langle C_i \rangle = \frac{1}{N} \sum_i C_i \) to quantify clustering.

- For a tree we have \( C = \langle C_i \rangle = 0 \) and for the complete graph \( C = \langle C_i \rangle = 1 \).
- Higher-order clustering coefficients can be defined similarly, using different subgraphs as basis.
5. Graph spectra

**Definition 5.8**

The **spectral density** of a graph $G = (V, E)$ is

$$\rho(\lambda) := \frac{1}{N} \sum_{i \in V} \delta(\lambda - \lambda_i)$$

where $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$ are the eigenvalues of the adjacency matrix $A$.

- **Perron-Frobenius**: $A$ has a real eigenvalue $\lambda_1 > 0$ with maximal modulus and real, non-negative eigenvector(s). If the graph is connected, it has multiplicity 1 and $|\lambda_j| < \lambda_1$ for all other eigenvalues with $j \neq 1$.

- For undirected graphs, $(A^n)_{ij}$ is equal to the **number of walks** (paths which allow repeated vertices) from $i$ to $j$ of length $n$. We also have

$$\text{Tr}(A^n) = \sum_{i=1}^{N} \lambda_i^n \quad \text{and} \quad \left(\text{Tr}(A)\right)^n = 0,$$

which can be used to derive statements like:

$$\sum_{i<j} \lambda_i \lambda_j = -|E|, \quad \sum_{i<j<l} \lambda_i \lambda_j \lambda_l = 2 \cdot \# \text{ of triangles in } G.$$
5. Graph Laplacian

**Definition 5.9**

The **Graph Laplacian** for a graph \((V, E)\) with adjacency matrix \(A\) is defined as

\[ Q := A - D \quad \text{where} \quad D = \left( \delta_{ij} \sum_{l \neq i} a_{il} : i, j \in V \right). \]

- \(Q\) has eigenvalues in \(\mathbb{C}\) with real part \(\text{Re}(\lambda) < 0\) except for \(\lambda_1 = 0\), which follows directly from the Gershgorin theorem and vanishing row sums.
- The **multiplicity of** \(\lambda_1\) **equals the number of connected components** in undirected graphs. Properly chosen orthogonal eigenvectors to \(\lambda_1\) have non-zero entries on the individual connected components.
- The smaller the second largest real part of an eigenvalue, the harder it is to cut \(G\) into separated components by removing edges.
- \(Q\) defines a generator matrix of a continuous-time random walk on \(V\) with transition rates \(a_{ij}\). Using weighted graphs, any finite state CTMC can be represented in this way.
- The Laplacian determines the first order linearized dynamics of many complex processes on graphs and is therefore of particular importance.
5. The Wigner semi-circle law

Theorem 5.1 (Wigner semi-circle law)

Let $A = (a_{ij} : i, j = 1, \ldots, N)$ be a real, symmetric matrix with iid entries $a_{ij}$ for $i \leq j$ with finite moments, and $\mathbb{E}[a_{ij}] = 0$, $\text{var}[a_{ij}] = \sigma^2$ (called a Wigner matrix). Then the spectral density $\rho_N$ of the matrix $A/\sqrt{N}$ converges in distribution to

$$
\rho_N(\lambda) \to \rho_{sc}(\lambda) := \begin{cases} 
(2\pi\sigma^2)^{-1} \sqrt{4\sigma^2 - \lambda^2}, & \text{if } |\lambda| < 2\sigma \\
0, & \text{otherwise}
\end{cases}.
$$

- The bulk of eigenvalues of unscaled Wigner matrices typically lies in the interval $[-2\sqrt{N}\sigma, 2\sqrt{N}\sigma]$.

- Adjacency matrices $A$ of $G_{N,p}$ random graphs are symmetric with iid $\text{Be}(p)$ entries with $\mathbb{E}[a_{ij}] = p$ and $\text{var}[a_{ij}] = p(1 - p)$, so are not Wigner matrices. $A$ has a maximal Perron-Frobenius eigenvalue of order $pN$, but all other eigenvalues have modulus of order $\sqrt{N}$.
  
  For fixed $p > 0$ or scaled $p = p_N \gg p_c = 1/N$ the Wigner semi-circle law holds for $N \to \infty$ with support width $4\sqrt{N}\sigma_N$.

  For $p = p_N \ll p_c = 1/N$ the asymptotic spectral density deviates from $\rho_{sc}$.

- There is a related circular law for non-symmetric Wigner matrices.
5. More general graphs and networks

- **For multigraphs**, multiple edges between nodes and loops \((a_{ii} > 0)\) are allowed.

- **Hypergraphs** \((V, E)\) are generalizations in which an edge can connect any number of vertices. Formally, the set of hyperedges \(E \subseteq \mathcal{P}(V)\) is a set of non-empty subsets of \(V\).

- In **bipartite graphs** the edge set can be partitioned into two sets \(V_1, V_2 \subseteq V\) each non-empty, with no connections within themselves, i.e. \(a_{ij} = a_{ji} = 0\) for all \(i, j \in V_1\) and also for all \(i, j \in V_2\).

  Simple undirected examples include regular lattices \(\mathbb{Z}^d\) for \(d \geq 1\) which are partitioned into sites with even and odd parity. Feed-forward neural networks are examples of directed graphs with bipartite or multi-partite structure.

- **Multilayer networks** \(M = (G, C)\) consist of a family of \(m\) (weighted or unweighted) graphs \(G_\alpha = (V_\alpha, E_\alpha)\) (called **layers** of \(M\)), and the set of interconnections between nodes of different layers

  \[
  C = \{ c_{\alpha,\beta} \subseteq V_\alpha \times V_\beta : \alpha, \beta \in \{1, \ldots, m\}, \alpha \neq \beta \}.
  \]

  Real examples include transportation networks or social networks with different types of connections.
6. E-R Random graphs

**Definition 6.1**

An *(Erdős-Rényi, short E-R)* random graph $G \sim \mathcal{G}_{N,K}$ has uniform distribution on the set of all undirected graphs with $N$ vertices and $K = |E|/2$ edges, i.e.

$$P_{N,k}[G = (V, E)] = \frac{1}{N(N-1)/2 \choose K}.$$ 

An *(E-R) random graph* $G \sim \mathcal{G}_{N,p}$ has $N$ vertices and each (undirected) edge is present independently with probability $p \in [0, 1]$, i.e.

$$P_{N,p}[G = (V, E)] = p^{|E|/2}(1 - p)^{N(N-1)/2 - |E|/2}.$$ 

- The ensemble $\mathcal{G}_{N,p}$ is easier to work with and is mostly used in practice, and for $N, K$ large, $\mathcal{G}_{N,K}$ is largely equivalent to $\mathcal{G}_{N,p}$ with $p = 2K/(N(N - 1))$.
- Since edges are present independently, graphs $G \in \mathcal{G}_{N,p}$ should typically be uncorrelated. Indeed, one can show that $\chi(G), \mathbb{E}[\chi] \to 0$ as $N \to \infty$. 

6. E-R Random graphs - properties

- The number of undirected edges for $G \sim G_{N,p}$ is random, $K \sim \text{Bi}\left(\frac{N(N-1)}{2}, p \right)$. For all $i$ by homogeneity, $k_i \sim \text{Bi}(N - 1, p)$ and $\mathbb{E}[\langle k \rangle] = \mathbb{E}[k_i] = (N-1)p$.

- The expected number of triangles in a $G_{N,p}$ graph is $\binom{N}{3}p^3$, and the number of triples is $\binom{N}{3}3p^2$. Since fluctuations are of lower order, this implies for all $G_N \sim G_{N,p}$

$$C(G_N) = \frac{3\binom{N}{3}p^3(1 + o(1))}{\binom{N}{3}3p^2(1 + o(1))} \to p \quad \text{as } N \to \infty .$$

- The expected degree distribution for $G_N \sim G_{N,p}$ is $\text{Bi}(N - 1, p)$. In the limit $N \to \infty$ with $p = p_N = z/(N - 1)$ keeping $z = \mathbb{E}[\langle k \rangle]$ fixed we have

$$\mathbb{E}[p(k)] = \mathbb{P}[k_i = k] = \binom{N - 1}{k}p_N^k(1 - p_N)^{N-1-k} \to \frac{z^k}{k!} e^{-z} .$$

Therefore, E-R $G_{N,p}$ graphs are sometimes called Poisson random graphs.

- In this scaling limit E-R graphs are locally tree-like, i.e. connected components

$$C_i^n := \{ j \in V : j \leftrightarrow i, d_{ij} \leq n \} , \quad n \text{ fixed}$$

are tree subgraphs as $N \to \infty$ with probability 1. Vertex degrees are $k_i \sim \text{Poi}(z)$ and iid $k_j \sim \text{Poi}(z) + 1$. 
6. Percolation

Percolation studies robustness of connectivity properties of graphs under deletion of edges or vertices (e.g. random attacks or immunization).

Definition 6.2

Consider a connected, undirected graph $G = (V, E)$. **Bond percolation** is a static probabilistic model with state space

$$S = \Omega = \{0, 1\}^E = \{e_{ij} \in \{0, 1\} : (i, j) \in E\},$$

and distribution $p = \mathbb{P}[e_{ij} = 1] = 1 - \mathbb{P}[e_{ij} = 0]$, i.e. $e_{ij} \sim \text{Be}(p)$ iid with $p \in [0, 1]$. Edges $(i, j) \in E$ are called **open** if $e_{ij} = 1$ and **closed** if $e_{ij} = 0$, and we denote by

$$G^o = (V, E^o) \quad \text{with} \quad E^o = \{(i, j) \in E : e_{ij} = 1\} \subseteq E$$

the (random) subgraph containing only open edges. A sequence of connected graphs $G_N$ of increasing size $|V_N| = N$ exhibits **percolation with parameter** $p$ if

$$|\bar{C}_N^o|/N \geq c > 0 \quad \text{as} \quad N \to \infty \quad \text{with probability 1},$$

where $|\bar{C}_N^o| = \max_{i \in V_N} |C_i^o|$ is the size of the largest connected component $\bar{C}_N^o$ of $G_N^o$. 
6. Percolation and E-R graphs

- Alternatively, percolation can be defined on an infinite graph \( G \) (e.g. \( \mathbb{Z}^d \)) with the **percolation probability**

\[
\theta(p) = \mathbb{P}[|C_0| = \infty] \begin{cases} = 0, & \text{for } p < p_c \\ > 0, & \text{for } p > p_c \end{cases},
\]

changing behaviour at a **critical value** \( p_c \in [0, 1] \).

- **Site percolation** vertices and their adjacent edges are deleted.

- E-R random graphs \( G_{N,p} \) have the same distribution as open subgraphs \( (G^o, E^o) \subseteq K_N \) under **percolation on the complete graph** \( K_N \) with parameter \( p \).

**Theorem 6.1 (Giant component for E-R graphs)**

Consider \( G_{N,p} \sim G_{N,p} \) with \( p = z/N \) and **maximal connected component** \( \bar{C}_{N,p} \). Then

\[
|\bar{C}_{N,p}| = \begin{cases} O(\log N), & \text{for } z < 1 \\ O(N^{2/3}), & \text{for } z = 1 \\ O(N), & \text{for } z > 1 \end{cases}
\]

and

\[
\theta(z) = \begin{cases} 0, & \text{for } z \leq 1 \\ > 0, & \text{for } z > 1 \\ \to 1, & \text{for } z \to \infty \end{cases}
\]

where \( \theta(z) := \lim_{N \to \infty} |\bar{C}_{N,p}|/N \) is a continuous, monotone increasing function of \( z \).

For \( z > 1 \), \( \bar{C}_{N,p} \) is the only **giant component**, and the second largest is \( O(\log N) \).

Local trees with \( 1 + \text{Poi}(z) \) degrees die out with probability 1 if and only if \( z \leq 1 \).
6. Preferential attachment

The prevalence of power-law degree distributions in real complex networks can be attributed to growth mechanisms subject to preferential attachment.

Definition 6.3

Starting with a complete graph \((V_0, E_0)\) of \(|V_0| = m_0\) nodes, at each time step \(t = 1, \ldots, N - m_0\) a new node \(j = t + m_0\) is added. It forms \(m \leq m_0\) undirected edges with existing nodes \(i \in V_{t-1}\) with a probability proportional to their degree

\[
\pi_{j \leftrightarrow i} = \frac{k_i}{\sum_{l \in V_t} k_l}
\]

(preferential attachment).

The resulting, undirected graph with \(N\) nodes and \(K = m_0(m_0 - 1)/2 + m(N - m_0)\) is called a Barabási-Albert random graph, denoted by \(G_{N,K}^{BA}\).

- As \(N \to \infty\), the average degree is \(\langle k \rangle = 2m\) and the degree distribution \(p_N(k)\) converges to a distribution \(p(k)\) with power law tail, i.e. \(p(k) = Ck^{-\alpha}\) for large \(k\) where \(\alpha = 3\), which is close to exponents observed for real-world networks. This is independent of the parameters \(m_0\) and \(m\).

- Characteristic path length and clustering coefficient typically behave like \(L = O(\log N)\) and \(C = O(N^{-0.75})\) for \(G_{N,K}^{BA}\) graphs, and they are uncorrelated.

- They are not homogeneous, the expected degree of nodes increases with age.
6. Preferential attachment

(A) power law for $\gamma = 1$, $m_0 = m = 5$, $N = 200K$, (B) exponential tail for $\gamma = 0$, $m_0 = m = 1, 3, 5, 7$, (C) degree increasing with time for $t_1 = 5$, $t_2 = 95$

taken from [A.-L. Barabási, R. Albert, Science 286(5439), 509-512 (1999)]

- Variations of the model connecting to vertices $i$ with probability proportional to $k_i + k_0$ lead to power law degree distributions with $\alpha = 3 + k_0/m$.

- For non-linear preferential attachment proportional to $k_i^\gamma$ we get

  $\gamma \in [0, 1)$: $p(k)$ has a stretched exponential tail $\exp(-Ck^{1-\gamma})$

  and the graph is assortative

  $\gamma > 1$: all vertices connect to $m$ super vertices and the graph is disassortative
6. Small-world networks

**Definition 6.4**

A sequence of connected graphs $G_N$ with increasing size $|V_N| = N$ exhibits the **small-world property**, if the characteristic path length $L(G_N) = O(\log N)$.

Examples include trees with degrees $k_i \geq 3$ and also the giant or largest component in E-R random graphs. In most graph models small-worldness is paired with low clustering coefficients, e.g. 0 for trees and $p$ for $G_{N,p}$ graphs. However, many real examples of small world networks exhibit also **large clustering coefficients**, such as networks of social contacts.

**Definition 6.5**

Consider a $2m$-regular ring graph with adjacency matrix $a_{ij} = \begin{cases} 1, & |i - j| \leq m \\ 0, & \text{otherwise} \end{cases}$ of size $N$ with a total number of $K = mN$ undirected edges. For all $i$, each edge $(i,j)$ with a clockwise neighbour with $j > i$ is **rewired** with probability $p \in [0, 1]$, i.e. replaced by an edge $(i, l)$ where $l$ is chosen uniformly among vertices not adjacent to $i$. The resulting graph is a **Watts-Strogatz random graph**, denoted by $G_{N,K}^{\text{WS}}$. 
6. Watts-Strogatz model

- W-S random graphs interpolate between a regular lattice for $p = 0$ and a $G_{N,K}$ E-R random graph conditioned on the event that all vertices have degree $k_i \geq m$.
- Expected clustering coefficient $\mathbb{E}[C(p)]$ and characteristic path length $\mathbb{E}[L(p)]$ are monotone decreasing functions of $p$ and show the following behaviour.

\[
N = 1000 \text{ and } m = 5, \text{ taken from } [D.J. \text{ Watts, S.H. Strogatz, Nature } \mathbf{393}, \text{ 440-442 (1998)}]
\]
Definition 6.6

The configuration model $G_{N,D}^{\text{conf}}$ is defined as the uniform distribution among all undirected graphs with $N$ vertices with a given degree sequence $D = (k_1, \ldots, k_N)$, such that $\sum_{i \in V} k_i = 2K$.

- Not all sequences $D$ that sum to an even number are graphical.
- Sampling is usually done by attaching $k_i$ half edges to each vertex $i$ and matching them randomly. This can lead to self loops and rejections.
- General randomized graphs with given degree distribution $p(k)$ can be sampled in the same way. If $k_{\text{max}} = \max_i k_i$ is bounded, one can show that these graphs exhibit a giant (connected) component of size $O(N)$ if
  \[ Q := \sum_{k \geq 0} k(k - 2)p(k) > 0, \]
  and if $Q < 0$ the largest component is of size $O(k_{\text{max}}^2 \log N)$.
- For directed versions with $D^{\text{in}}$ and $D^{\text{out}}$ we need $\sum_{i \in V} k_i^{\text{in}} = \sum_{i \in V} k_i^{\text{out}}$. 
6. Planar graphs and spatial point processes

Definition 6.7

A **planar graph** is an undirected graph that can be embedded in the plane, i.e. it can be drawn in such a way that no edges cross each other. The edges of a particular embedding partition the plane into **faces**. A connected planar graph $G$ has a **dual graph** $G^*$, which has one vertex in each face of $G$, and a unique edge crossing each edge of $G$. $G^*$ may be a multigraph with self-loops.

A **maximal planar graph** is called a **triangulation**.

- Every planar graph is 4-partite or 4-colourable.
- In a triangulation each face is bounded by three edges. By induction, every triangulation with $N > 2$ nodes has $K = 3N - 6$ undirected edges and $2N - 4$ faces.

Definition 6.8

A random countable set $\Pi \subseteq \mathbb{R}^d$ is called a **spatial point process**. $\Pi \subseteq \mathbb{R}^d$ is called a homogeneous **Poisson point process** $\text{PPP}(\lambda)$ with rate $\lambda > 0$ if

- for all $A \subseteq \mathbb{R}^d$ we have $N(A) := |\Pi \cap A| \sim \text{Poi}(\lambda|A|)$,
- for all disjoint $A_1, \ldots, A_n \subseteq \mathbb{R}^d$, $N(A_1), \ldots, N(A_n)$ are independent.
6. Planar graphs and spatial point processes

- To sample from a PPP(\( \lambda \)) e.g. in a box \( A = [0, L]^d \), pick \( N(A) \sim \text{Poi}(\lambda L^d) \), then place \( N(A) \) particles independently in \( A \) each with uniform distribution, i.e. pick the \( d \) coordinates uniformly in \([0, L]\).

- A Poisson process \( \text{PP}(\lambda) \) is equivalent to a PPP(\( \lambda \)) on \([0, \infty)\).

**Definition 6.9**

Let \( \Pi = \{x_1, x_2, \ldots \} \) be a countable subset of \( \mathbb{R}^d \), endowed with a distance function \( d(x, y) \). A **Voronoi tessellation (or diagram)** is given by the family of **Voronoi cells** \( \{A_1, A_2, \ldots \} \subseteq \mathbb{R}^d \) where

\[
A_i = \{ x \in \mathbb{R}^d : d(x, x_i) \leq d(x, x_j) \text{ for all } j \neq i \}
\]

is the set of points closest to \( x_i \).

**Properties in 2 dimensions.**

- The shape of Voronoi cells depends on the distance function, for Euclidean distance \( d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \) they are convex polygons, and boundaries between adjacent cells are straight lines.

- The dual graph of a Voronoi diagram of a set \( \Pi \) is called **Delaunay triangulation**, which is not unique if 4 or more cells intersect in a point.