Mechanism of the Drift of a Spiral Wave in an Inhomogeneous Medium

E.A. Ermankova and A.M. Pertsov – translated by Iliana Peneva

(Dated: Wednesday 1st February, 2017)

Pertsov, A.M. and Ermakova, E.A., “Mechanism of the drift of a spiral wave in an inhomogeneous medium”. Biofizika, 33(2): 338-342, 1988.

In inhomogeneous active medium, refractory spiral wave is not stationary: on the contrary, it moves, drift. The first time this drift was shown to exist was in the analysis of model [1] (Krinsky). After that the movement of spiral waves (reverberator) in other models [2-4].

In this work we analyse the mechanism of drift in a weakly inhomogeneous medium, the resulting analytic description of the drift is applicable in situations where the period of rotation of the reverberators is much greater than the refractory period of the medium ($T/T_r \gg 1$).

The equation of the trajectory of the drift of the reverberator in inhomogeneous medium

As is known, in a homogeneous medium near the centre of the reverberator [1], wave excitations move evenly around a circle with some radius R (Fig. 1a). In inhomogeneous medium this trajectory stops being circular (Fig. 1b). To describe this trajectory, we use equations for the curvature of a plane curve [6]:

$$k = (\dot{x}\ddot{y} - \dot{y}\ddot{x})/(\dot{x}^2 + \dot{y}^2)^{3/2}. \tag{1}$$

We will write

$$k = 1/R(x, y), \tag{2}$$

which means that the trajectory at each point of the inhomogeneous medium is inversely proportional to the instantaneous radius of the reverberator ($R(x, y)$ is the radius of the reverberator in a homogeneous medium with parameters which correspond to coordinates x, y). Then using (1) and (2), with properly chosen initial conditions, we can determine uniquely the trajectory.

Calculation of the trajectory in medium with weak inhomogeneity

We will show that in a sufficiently weak inhomogeneous medium the trajectory of the drift is close to a cycloid (the equation has a harmonic component and a dissipative term which determines the speed of the drift). We will find the solution in the neighbourhood of some random points P. To facilitate we will assume that the gradient of $R(x, y)$ is along the y-axis [2]. Then eqn. (1) can be written to order ϵ as

$$k = k_0 + \epsilon y = (\dot{x}\ddot{y} - \dot{y}\ddot{x})/(\dot{x}^2 + \dot{y}^2)^{3/2}, \tag{3}$$

where $\epsilon \ll 1$ and characterises how inhomogeneous the medium is

$$\epsilon = k'_y = -R'_y R^{-2}(P). \tag{4}$$

To solve (3), we use the method of perturbations. When we take $\epsilon = 0$, we get the solution for a homogeneous medium

$$x_0 = R_0 \cos(\omega t), \qquad y_0 = R_0 \sin(\omega t), \tag{5}$$

where R_0 is the radius and (x_0, y_0) is the centre. **GPA: check here** where $\omega = 2\pi/T$, T [something] at P. Then, taking into account ϵ to first order, we get

$$x = x_0 + \epsilon x_1, \qquad y = y_0 + \epsilon y_1. \tag{6}$$

Subbing (6) into (3) and equating terms order by order in ϵ we get

$$\ddot{x}_1 + 2\omega\dot{y}_1 = 0, \qquad \ddot{y}_1 - 2\omega\dot{x}_1 = R_0^3\omega^2. \tag{7}$$

The solutions are

$$\begin{aligned} x_1 &= C_1 \frac{1}{2\omega} \sin(2\omega t) + C_2 \frac{1}{2\omega} \cos(2\omega t) + \frac{1}{2} R_0^3 \omega t + C_3, \\ y_1 &= -C_1 \frac{1}{2\omega} \cos(2\omega t) + C_2 \frac{1}{2\omega} \sin(2\omega t) + \frac{1}{2} R_0^3 \omega t + C_4. \end{aligned} \tag{8}$$

The dissipative term is proportional to t, (in the x_1 eqn), which means that, to a first approximation, the drift is perpediuclar to the inhomogeneity gradient. The coefficient of t, multiplied by ϵ, determines the speed of the drift v:

$$v = \frac{1}{2} \epsilon R_0^3 \omega. \tag{9}$$

The direction of the drift depends on the sign of ω.

The periodic and constant terms in (8) determine the difference between the perturbed and unperturbed solutions, *i.e.* the size and the shape of the core, depending on the initial conditions. When you choose the initial conditions appropriately, your periodic and constant terms are equal to 0, and the equations of the tip of the reverberator are the same as those for a cycloid

$$x = R_0 \cos(\omega t) + \frac{1}{2} \epsilon R_0^3 \omega t, \qquad y = R_0 \sin(\omega t). \tag{10}$$

We can show that if we account for second order terms (in ϵ), it won't result in drift along the the gradient. The physical mechanism of the drift can be illustrated in an inhomogneous medium, which consists of two homogeneous half planes, with differing core radii − $R = R_1$ when $y > 0$, $R = R_2$ when $y \leq 0$ ($R_1 > R_2$). Thus the gradient of R is directed up. At the beginning the tip of the reverberator is positioned on the border ($y = 0$). We can easily see that its trajectory is composed of half circles of radius R_1 and R_2 and the drift is only along the x-axis perpendicular to ∇R [3], with speed $v = 4(R_1 - R_2)/(T_1 + T_2)$, and the sign of v changes with the sign of ω.

Comparison with results from numerical experiments

As the numerical experiments of the FitzHugh-Nagumo model [7]. This means that the phenomenological theory describes well enough the drift in a medium where the period of the reverberator is much bigger than the refractory period ($T \gg T_r$). In Fig. 2a, we can see the drift of the reverberator in regions where $T/T_r = 7$. The inhomogeneity in parameter g_f is responsible for the excitability of the medium, $dg_f/dy = -0.001$, which corresponds to $dR/dy > 0$. It is obvious that the drift is almost perpendicular to the gradient in the direction of increase in x – it appears when $\omega > 0$ and $dR/dy > 0$. Comparison between the speed of the drift in numerical experiments and the analytic solution is presented in figure 2b. We solve analytically (9) using (4), and get

$$v = -\pi \frac{dR}{dy}\frac{R}{T} = -\pi \frac{dR}{dg_f}\frac{dg_f}{dy}\frac{R(g_f)}{T(g_f)}. \tag{11}$$

The values $R(g_f)$, $T(g_f)$, dR/dg_f are taken from numerical experiments in a homogeneous medium. It is obvious that the speed from numerics and analytics agree well. Their relative difference is 15%. **GPA: check** something about 10%.

Overlap with existing theory

As we already noted, we saw there is relatively good agreement between theory and experiment when $T/T_r \gg 1$. The differences when we decrease this value have a simple physical interpretation. In Fig. 3 there is a small section of the trajectory of the spiral wave. The shaded region is covered by the excitations for one period of rotation. The not shaded part is the unexcited part of the medium (the core). In the core the amplitude of excitations can be quite small, and for some parameters a considerable part of the core can be in a resting state [8, 9]. It is obvious that excitations alternate in the shaded and unshaded regions. So if the period of rotation is not significantly larger than the period of restoration of the medium (T_r), the medium doesn't succeed to forget the previous excitations, and the conditions in these regions become different. As a result, there is an additional (depending on the prehistory, T/T_r) disturbance which is not accounted for by (1), (2).

We have developed a theory of this, which gives good results only when $T/T_r \gg 1$.

Studying the time of restoration. Longitudinal drift.

As follows from above (Fig. 3), the disturbances depending on the prehistory appear periodically, and more precisely resonate with period equal to the period of rotation of the spiral wave. It is not difficult to show that this result leads to the appearance of the longitudinal drift. We account for this new effect with

$$k = k_0 + \epsilon y + \epsilon\varphi(t) = (\dot{x}\ddot{y} - \dot{y}\ddot{x})/(\dot{x}^2 + \dot{y}^2)^{3/2}, \tag{12}$$

where $\varphi(t)$ is a periodic function, and $\epsilon \ll 1$. The resonant character of these disturbances allow us to analyse their influence on the drift, even when we don't have information about $\varphi(t)$.

GPA: something about the form of φ: $\varphi(t) = \alpha\sin(\omega t) + \beta\cos(\omega t)$. We use the method of perturbation with small parameters, and consider only the first harmonics in ω, and we can show that in case $\beta \neq 0$, there is a component along the y-axis (along the gradient of the inhomogeneous medium). In this case the increments x_1 and y_1 have the following form: $x_1 = \frac{1}{2}R_0^2\omega(\alpha + R_0)t$ and $y_1 = -\frac{1}{2}R_0^2\beta\omega t$. y_1 is not periodic, which leads to drift along the y-axis.

From this it follows that if $\beta \neq 0$, the drift will be at an angle to the gradient of the inhomogeneous medium, which agrees with earlier results in [4], which studies the drift in inhomogeneous medium with constant gradient. Unlike v_x, we cannot derive an explicit expression for β, but it obviously depends on the time of the restoration of the medium – numerical experiments have shown that when $T_r \sim T$ the drift along the gradient dominates – we should note that the longitudinal drift does not depend on the sign of rotation of the reverberator, which is the sign of ω. Further, apparently, when $T_r/T \to 0$, $\beta \to 0$. Assuming $\beta \approx bT_r/T = \frac{2\pi}{\omega}bT_r$ (b constant), $y_1 = -bR_0^2T_rt$, independent of the sign of ω.

Discussion

As the experiments have shown, the spiral wave in active medium have different nature. In the regions of parameters which are close to the border there are autowaves, step spirals λ – as a consequence the radius of the cores ($R \approx \lambda/2\pi$) become more sensitive to changes in the parameters. (????) It is shown in numerical experiments with the FitzHugh-Nagumo model [7], and chemically active medium Belousov-Zhabotinsky [10]. Similar results are shown in experiments with hearts [11]. From this it follows that experiments in the described regions of the parameters, there could be a large ∇R, and high speed of the drift of the spiral waves could be expected as well.

When we get close to the border, stable autowaves appear and T/T_r increases. In the heart when T/T_r increases, we can observe the suppression of incoming sodium current, which determines the excitability of the heart cells [11]. For chemically active medium in the BZ, a similar thing appears in the regions with lower concentration of sulphuric acid and ferrin, where the experiments show that $T/T_r > 2$ [10]. Near the border stable autowaves appear together with increase in the speed of the drift which appears in the direction perpendicular to the gradient.

Figure captions

Fig. 1: The central region of the reverberator in (a) homogeneous (b) inhomogeneous medium (in the FN model, numerical experiment). $g_f = 0.78$. The shaded "excited" area, $E/E_{\max} \geq 0.5$, and the arrows show the direction of the waves around the rotation core. (b) is the reverberator in a medium consisting of two half planes with $g_f = 0.73$ (in the upper half plane) and $g_f = 1.3$ (in the lower half plane). 1-5 are the consecutive locations of the waves. The path shown is the tip of the spirals. We can see that the excited area is becoming bigger in the lower part of the medium (location 5) which is caused by the inhomogeneity.

Fig. 2: Drift of the reverberator in inhomogeneous medium. (a) Consecutive locations of the centre of the reverberator. 1, 5, 10, 14 corresponds to the number of periods executed. $T/T_r \approx 7$. $dg_f/dy = -0.001$, $g_f = 0.73 - 0.77$, $t_1 = 0.5$, (b) comparison between the speed of the drift using (11) and the points are the numerical experiments. ($dg_f/dy = -0.0018$)

Fig. 3: To analyse the role of the refraction. 1 and 2 are the trajectory of the tip of the spiral for two consecutive periods of the drift of the reverberator. The shaded area is swept wave during the first period.

[1] We do not consider the case of non stationary reverberator [5] in this work.

[2] Translator's note: maybe . . . Iliana

[3] The speed of the drift is in the direction perpendicular to ∇R also when ∇R is not small – the function $R(x, y)$ is monotone along the gradient of inhomogeneity – the corresponding theorem is proved in [7].