Umbilics in orientational order

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March 7, 2016
Described by a director field, $n(r)$.

$|n| = 1$, $n \sim -n$, i.e $n \in \mathbb{R}P^2$. 

The orientational order in liquid crystals represents the breaking of a continuous symmetry. As a consequence, global rotations of the director do not cost any energy, but local rotations, leading to gradients in the director field, do [3]. There are three independent distortions of the director field in a nematic liquid crystal, known as splay, twist and bend [1–5] and shown in Fig. 1.2. In terms of derivatives of the director field these three distortions correspond to the
FIG. 5. M.B.B.A. additionné d'un faible pourcentage de benzoate de cholestérol, examiné entre verres frottés... figures, voir le texte. Les échelles sont pratiquement les mêmes que pour la figure 1. Chaque segment représente 20 µm.
Energetics of Liquid Crystals

- Free energy naturally described by the geometry of the system - gradients of the director.

\[
F = \int \left( \frac{K_1}{2} (\nabla \cdot n)^2 + \frac{K_2}{2} (n \cdot \nabla \times n + q_0)^2 + \frac{K_3}{2} ((n \cdot \nabla)n)^2 \right)
\]

\(F\) \(n \cdot \nabla \cdot n \neq 0\) \(n \cdot \nabla \times n \neq 0\) \(- (n \cdot \nabla)n \neq 0\)

Figure 1.2: Schematic illustration of the elastic distortions in a nematic liquid crystal.
People have thought about textures in liquid crystals etc. by thinking about defects.
Classification of defects

Homotopy Theory

- Classifying **continuous maps** from real space into order parameter space, using homotopy theory.

\[ \pi_1(\mathbb{R}P^2) \simeq \mathbb{Z}_2 \]

\[ \pi_2(\mathbb{R}P^2) \simeq \mathbb{Z} \]
Non trivial textures with no defects
The Hopf fibration

FIG. 2 (color). (a) A simulation of a toron in which the point hedgehogs are replaced with disclination loops. (b) Flow lines of the famous Hopf fibration. (c) The preimage surface of the Hopf fibration, which one can get from (a) by bringing together the two disclination loops through the center of the spool. (d) An experimental image of the same texture.

- Discontinuities don’t tell us everything. Here’s a topologically non trivial texture containing no defects.
- \( \pi_3(\mathbb{R} P^2) \cong \mathbb{Z} \)
Non trivial textures with no defects

Lambda lines

▶ Not singularities in the director - degeneracies in the derivatives. Umbilic lines.
Using the differential structure

▶ The texture is more than continuous - it's differentiable.
▶ The derivatives have obvious structural importance

\[ F = \int \left( \frac{K_1}{2} (\nabla \cdot n)^2 + \frac{K_2}{2} (n \cdot \nabla \times n + q_0)^2 + \frac{K_3}{2} ((n \cdot \nabla)n)^2 \right) \]

▶ Rather than looking at \( n(r) \) alone, also look at \( \nabla n(r) \). These gradients contain structural, topological information.
The structure of $\nabla n$

- At each point $n(r)$ defines a line and a plane - it splits the tangent bundle.
  $$T\mathbb{R}^3 \simeq L_n \oplus \xi$$

- $\nabla n$ eats an element of $T\mathbb{R}^3$, and returns a direction in the orthogonal plane $\xi$
  $$\nabla n \in \Gamma(T^*\mathbb{R}^3 \otimes \xi) \simeq \Gamma((L^*_n \otimes \xi) \oplus (\xi^* \otimes \xi))$$
\(\nabla n\) in a local frame

- We see this explicitly by expressing \(\nabla n\) in a nice local frame. Let \(\{d_1, d_2\} \in \xi\). Then \(\{n, d_1, d_2\}\) form a local basis.

- Two parts - bend (derivatives in the direction of \(n\)), and everything else.

\[
\nabla n \in \Gamma(T^*\mathbb{R}^3 \otimes \xi) \simeq \Gamma((L_n^* \otimes \xi) \oplus (\xi^* \otimes \xi))
\]

\[
\begin{pmatrix}
0 & (n \cdot \nabla)n \cdot d_1 & (n \cdot \nabla)n \cdot d_2 \\
0 & (d_1 \cdot \nabla)n \cdot d_1 & (d_1 \cdot \nabla)n \cdot d_2 \\
0 & (d_2 \cdot \nabla)n \cdot d_1 & (d_2 \cdot \nabla)n \cdot d_2
\end{pmatrix}
\]
Umbilic Lines

Let's focus on the part that lives in \((\xi^* \otimes \xi)\). Call it \(\nabla_{\perp} n\). It's analogous to the shape operator.

The action of \(SO(2)\) on \((\xi^* \otimes \xi)\) gives a natural way to further decompose \(\nabla_{\perp} n\).

\[
\nabla_{\perp} n = \frac{1}{2} \nabla \cdot n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} n \cdot \nabla \times n \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} \Delta_1 & \Delta_2 \\ \Delta_2 & -\Delta_1 \end{pmatrix} \begin{pmatrix} \xi^* \otimes \xi \end{pmatrix} = I \oplus J \oplus E
\]

The places where \(\Delta \in \Gamma(E)\) vanishes are called Umbilic Lines. These turn out to contain structural information about the vector bundle \(\xi\).

\(\Delta\) really does vanish along lines. 2 constraints in \(\mathbb{R}^3\).
Zeroes and Topology

- Let’s motivate why looking at the zeroes of $\Delta$ is interesting.
- An example of the connections between
  - zeroes of a section of a vector bundle
  - structure of a vector bundle
  - topology of a space

Is the Hopf index theorem.
Hopf Index Theorem
aka "the hairy ball theorem"

\[ \sum \text{index}_{x_i}(\nu) = \chi(M) \]

- We cannot construct a nonzero section of \( T(S^2) \).
- **Why** we can’t is related to \( \chi(S^2) \), a purely topological thing.
Above we had a surface, $S^2$, and its tangent bundle $T S^2$. Analogous constructions exist for a general vector bundle over a general space. In particular, for the bundle $\xi$ (and $(\xi^* \otimes \xi)$) defined by the director field. We learn structural information about the bundle $\xi$ by studying the zeroes of $\Delta$.

2.2 Vector Fields on Surfaces

It is well-known that vector fields on the sphere always have zeros; it is even well-known that if you add up the indices of the zeros you get 2, the Euler characteristic for the sphere. Sometimes this is called the 'hairy ball theorem'; more properly it is the Poincaré-Hopf index theorem.

To understand this we need to know what is meant by the index of a zero of a vector field. A quick look at some examples makes the idea clear enough; the index is how many times the vector rotates as you go around a little circle about the zero. Everywhere on this little circle the vector is non-zero, so you can scale it to have unit length. Then it too is a point on a circle and the behaviour of the vector field around the zero is characterised by a map from a circle to a circle, and so by the degree, or winding number, of that map. Again, this is really about orientation. In order to know whether the vector field is winding positively or negatively we need a local orientation for the vector bundle (both base and fibre) around the zero. And in order to compare, or add up, the indices of two or more zeros we need to extend this consistently to a global orientation. That given, computing the sum of the indices of the zeros of a vector field is relatively easy, and quite enjoyable.

It is useful to see why there are always zeros on the sphere by trying explicitly to make a vector field without any. Here is one such construction: start at the south pole with some non-zero vector and move away in all directions trying to hold the vector fixed as you go. There are two things to straighten out here. What we mean by 'move away' is 'travel along a geodesic passing through the south pole', and what we mean by 'hold the vector fixed' is 'carry it by parallel transport'.

This strategy for generating sections is remarkably enlightening; it extends the vector field uniquely to be non-zero everywhere except where the geodesics reconverge, i.e. at the conjugate points. For the sphere the geodesics reconverge only at the north pole, so we need only look at the behaviour around that point. Applying the parallel transport rule we see that the vector field on a little circle around the north pole has index +2, so that the north pole itself must be a zero. Of course, the same construction can be applied to all other closed surfaces, both orientable and non-orientable, and it is quite enjoyable to do so.

2.3 The Zero Locus

Given any vector bundle, and any section of it, one can ask: "Where is the section zero?" This will be a subset of the base; it is called the zero locus. In our preceding examples of line bundles over the circle, or vector fields on surfaces, this was a discrete set of points. For the normal bundle to a surface the zero locus was a set of one-dimensional curves. In all cases the zeros were either of different types – they came with an index – or were to be counted only modulo 2.

The exact nature of the zero locus depends on the exact section we are considering. Any alterations in the section will be reflected in changes to the zero locus; its location on the base, the number of separate components, and so forth. So none of these things capture the nature of the vector bundle. An important property that does is that the zero locus is a cycle, by which we mean it has no boundary. This is kind of trivial when the zero locus is a set of points, for...
What I’m doing

- \( \nabla_\perp n \in \Gamma(\xi^* \otimes \xi) \) has been partially analysed.
- I’m beginning work on similar constructions for \((L_n^* \otimes \xi)\), and seeing how they relate to one another.

\[
\nabla n \in \Gamma(T^*\mathbb{R}^3 \otimes \xi) \cong \Gamma((L_n^* \otimes \xi) \oplus (\xi^* \otimes \xi))
\]

\[
\begin{pmatrix}
0 & (n \cdot \nabla)n \cdot d_1 & (n \cdot \nabla)n \cdot d_2 \\
0 & (d_1 \cdot \nabla)n \cdot d_1 & (d_1 \cdot \nabla)n \cdot d_2 \\
0 & (d_2 \cdot \nabla)n \cdot d_1 & (d_2 \cdot \nabla)n \cdot d_2
\end{pmatrix}
\]