HW-LGM-AnalyticSwapsCapFloorsOnRFR

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Contents

1 Reminder about pricing under Q and Q^T measures

According to the definition of Q-measure for first equaity (and Bayes formula for second), we have:

$$\boxed{PV_t = B_t \mathbb{E}_t^Q \left[\frac{X_T}{B_T} \right] = p(t, T) \mathbb{E}_t^{Q^T} \left[X_T \right]} \tag{1}$$

Indeed, according to Bayes formula, and since

$$\frac{dQ}{dQ^T}(t) = \frac{B_t}{p(t,T)} \cdot \frac{p(0,T)}{B_0},$$

we obtain:

$$PV_t = B_t \mathbb{E}^Q \left[\frac{X_T}{B_T} | F_t \right] = B_t \frac{\mathbb{E}_t^{Q^T} \left[\frac{dQ}{dQ^T}(T) \frac{X_T}{B_T} | F_t \right]}{\mathbb{E}_t^{Q^T} \left[\frac{dQ}{dQ^T}(T) | F_t \right]} =$$

$$= B_t \frac{\mathbb{E}_t^{Q^T} \left[\frac{B_T \cdot p(0,T)}{p(T,T) \cdot B_0} \frac{X_T}{B_T} | F_t \right]}{\mathbb{E}_t^{Q^T} \left[\frac{B_T \cdot p(0,T)}{p(T,T) \cdot B_0} | F_t \right]} = B_t \frac{p(0,T) \mathbb{E}_t^{Q^T} \left[X_T | F_t \right]}{p(0,T) \mathbb{E}_t^{Q^T} \left[\frac{B_T}{p(T,T)} | F_t \right]} = B_t \frac{\mathbb{E}_t^{Q^T} \left[X_T | F_t \right]}{\frac{B_t}{p(t,T)}} = p(t,T) \mathbb{E}_t^{Q^T} \left[X_T | F_t \right],$$

where we used that $B_0 = 1, p(T,T) = 1$ and $\mathbb{E}_t^{Q^T} \left[\frac{B_T}{p(T,T)} | F_t \right] = \frac{B_t}{p(t,T)}$ due to definition of T-forward measure.

For $t = 0$ we have a less difficult proof:

$$PV_0 = B_0 \mathbb{E}^Q \left[\frac{X_T}{B_T} \right] = B_0 \mathbb{E}^{Q^T} \left[\frac{dQ}{dQ^T}(T) \frac{X_T}{B_T} \right] = B_0 \mathbb{E}^{Q^T} \left[\frac{B_T \cdot p(0,T)}{p(T,T) \cdot B_0} \frac{X_T}{B_T} \right] = p(0,T) \mathbb{E}^{Q^T} \left[X_T \right]$$

2 Reminder about formulas for Libor IRS and Caps

Recall that by definition of Libor rate we have:

$$L(T, T+\tau) = L(T, T, T+\tau) = \frac{1}{\tau} \left(\frac{1}{p(T, T+\tau)} - 1 \right)$$

Forward Libor rate (predicted at time t) is by definition:

$$F(t, T, T+\tau) = \frac{1}{\tau} \left(\frac{p(t,T)}{p(t, T+\tau)} - 1 \right)$$

Since rhs is some tradable portfolio, divided by the price of $T+\tau$-bond, it is a martingale under $T+\tau$-forward measure, and hence, rhs (that is $F(t,T,T+\tau)$) is also a martingale under $T+\tau$-forward measure. Hence

$$\boxed{F(t,T,T+\tau) = \mathbb{E}^{T+\tau} \left[F(T,T,T+\tau) | F_t \right] = \mathbb{E}^{T+\tau} \left[L(T,T,T+\tau) | F_t \right]} \quad (2)$$

3 Proposed approximation for compounding and simple average

Now consider the payoffs of IRS and Cap on Libor rate with (constant over all periods) notional N and n periods $\tau_i = T_i - T_{i-1}, T_0 = 0$

According to the second part of (1) and (2) , PV of IRS is given by the formula

$$PV_t^{swap} = \sum_{i=1}^{N} p(t,T_i)\mathbb{E}_t^{T_i}\left[L(T_{i-1},T_i) - K\right] N\tau_i = \sum_{i=1}^{N} p(t,T_i)\mathbb{E}_t^{T_i}\left[F(t,T_{i-1},T_i) - K\right] N\tau_i \tag{3}$$

$$PV_t^{cap} = \sum_{i=1}^{N} p(t,T_i)\mathbb{E}_t^{T_i}\left[L(T_{i-1},T_i) - K\right]^{+} N\tau_i$$

Now remember that we are considering IRS and Cap on RFR, so instead of $L(T_{i-1},T_i)\cdot N\cdot\tau_i$ floating leg will pay $A(T_{i-1},T_i)\cdot N\cdot\tau_i$ in case of Simple Average and $R(T_{i-1},T_i)\cdot N\cdot\tau_i$ in case of Compounding, where $A(T_{i-1},T_i)$ and $R(T_{i-1},T_i)$ are defined as following:

$$A(T_{i-1},T_i) = \frac{1}{\tau_i}\left[\sum_{k=1}^{n}\tau_{i_k} r_{t_{i_k}}\right] \sim \frac{1}{\tau_i}\left[\int_{T_{i-1}}^{T_i} r_u du\right]$$

$$R(T_{i-1},T_i) = \frac{1}{\tau_i}\left[\prod_{k=1}^{n}(1+\tau_{i_k} r_{t_{i_k}}) - 1\right] \sim \frac{1}{\tau_i}\left[e^{\int_{T_{i-1}}^{T_i} r_u du} - 1\right]$$

According to IRS pricing formula (3), now we need to find conditional expectations of $A(T_{i-1},T_i)$ and $R(T_{i-1},T_i)$ in T_i-forward measure. Let's denote them $R_i(t)$ and $A_i(t)$:

$$R_i(t) := \mathbb{E}^{T_i}\left[R(T_{i-1},T_i)|F_t\right] = \frac{1}{\tau_i}\left(\frac{p(t,T_{i-1})}{p(t,T_i)} - 1\right)$$

$$A_i(t) := \mathbb{E}^{T_i}\left[A(T_{i-1},T_i)|F_t\right]$$

$R_i(t)$ and $A_i(t)$ does not admit a model-free expression, so we will assume Hull-White (and then LGM) model for interest rates.

In both cases $\int_{T_{i-1}}^{T_i} r_u du$ will have Normal distribution with parameters μ and σ^2, and hence we can compute $R_i(t)$ and $A_i(t)$ and so we can easily compute prices for IRS and Cap for Simple Average and Compounding:

For IRS:

$$PV_t^{swap,Cmp} = \sum_{i=1}^{N} p(t,T_i)\mathbb{E}_t^{T_i}\left[R(T_{i-1},T_i) - K\right] N\tau_i = \sum_{i=1}^{N} p(t,T_i)\left[R_i(t) - K\right] N\tau_i$$

$$PV_t^{swap,SA} = \sum_{i=1}^{N} p(t,T_i)\mathbb{E}_t^{T_i}\left[A(T_{i-1},T_i) - K\right] N\tau_i = \sum_{i=1}^{N} p(t,T_i)\left[A_i(t) - K\right] N\tau_i$$

For Cap:

$$PV_t^{cap,Cmp} = \sum_{i=1}^{N} p(t,T_i)\mathbb{E}_t^{T_i}\left[R(T_{i-1},T_i)-K\right]^+ N\tau_i = \sum_{i=1}^{N} p(t,T_i)\mathbb{E}_t^{T_i}\left[\tau_i R(T_{i-1},T_i)-\tau_i K\right]^+ N =$$

$$= \sum_{i=1}^{N} p(t,T_i) N Black\left(1+\tau_i R_i(t), 1+\tau_i K, Var_t^{T_i}\left[\int_{T_{i-1}}^{T_i} r_u du\right]\right) =$$

$$\boxed{\sum_{i=1}^{N} p(t,T_i) N\left[(1+\tau_i R_i(t))\Phi(d_1)-(1+\tau_i K)\Phi(d_2)\right]},$$

where

$$d_1 = \frac{\ln\frac{1+\tau_i R_i(t)}{1+\tau_i K} + \frac{1}{2} Var_t^{T_i}\left[\int_{T_{i-1}}^{T_i} r_u du\right]}{\sqrt{Var_t^{T_i}\left[\int_{T_{i-1}}^{T_i} r_u du\right]}}$$

$$PV_t^{cap,SA} = \sum_{i=1}^{N} p(t,T_i)\mathbb{E}_t^{T_i}\left[A(T_{i-1},T_i)-K\right]^+ N\tau_i = \sum_{i=1}^{N} p(t,T_i)\mathbb{E}_t^{T_i}\left[\tau_i A(T_{i-1},T_i)-\tau_i K\right]^+ N =$$

$$= \boxed{\sum_{i=1}^{N} p(t,T_i) N\left[(\tau_i A_i(t)-\tau_i K)\Phi(d) + \sqrt{Var_t^{T_i}\left[\int_{T_{i-1}}^{T_i} r_u du\right]}\phi(d)\right]},$$

where

$$d = \frac{\tau_i A_i(t)-\tau_i K}{\sqrt{Var_t^{T_i}\left[\int_{T_{i-1}}^{T_i} r_u du\right]}}$$

So what we are left to do is to compute conditional expectation and conditional variance of $\int)T_{i-1}{}^{T_i}$ in T_i -forward measure (and in Q-measure also, just for an exercise) in Hull-White and LGM models.

4 Hull-White model

4.1 HW: model setup

For more details see page 73 of Brigo-Mercurio.
Hull and White (1994a) assumed that the instantaneous short-rate process evolves under the risk-neutral measure according to:

$$dr_t = (\theta_t - ar_t)dt + \sigma dW_t, \tag{4}$$

where a and σ are positive constants and θ is chosen so as to exactly fit the term structure of interest rates being currently observed in the market (see below how the formula for θ is obtained. As the input to calibrate this model, the initial forward curve $f^M(0,T)$ is given. Here $f^M(0,T)$ is market instantaneous forward rate at time 0 for the maturity T , i.e.,

$$f^M(0,T) = -\frac{\partial p^M(0,T)}{\partial T},$$

where $p^M(0,T)$ are the market discount factor for the maturity T.
Hull-White SDE (4) admits exact solution (since it is an Ornshtein-Uhlenbeck process), but for calibration of θ_t (and for subsequent comparison with LGM model) it is more convenient to split r_t into stochastic part x_t and deterministic part α_t (process x_t now oscillates around zero, not for initial forward curve, as it is in case of r_t):

$$r_t = x_t + \alpha_t$$

Hence, plugging in (4) that $r_t = x_t + \alpha_t$, we obtain:

$$dx_t + d\alpha_t = dr_t = (\theta_t - a(x_t + \alpha_t))dt + \sigma dW_t = [-ax_t dt + \sigma dW_t] + (\theta_t - a\alpha_t)dt$$

Hence we obtain SDEs for x_t and α_t:

$$\begin{cases} dx_t = -ax_t dt + \sigma dW_t, x_0 = 0 \\ d\alpha_t = (\theta_t - a\alpha_t)dt, \alpha_0 = 0(?) \end{cases}$$

Now we will show, how to calibrate this model to initial market curve $f^M(0,t)$: we will obtain explicit formulas for x_t, α_t, θ_t, and they will be the following:

$$\begin{cases} x_t = x_0 + \sigma \int_0^t e^{-a(t-u)} dW_u, \\ \alpha_t = f^M(0,t) + \frac{\sigma^2}{2a^2}\left(1 - e^{-at}\right)^2, \\ \theta_t = \frac{\partial f^M(0,t)}{\partial t} + af^M(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right) \end{cases}$$

4.2 HW: integrating SDE

a) Finding x_t is just solving Ornstein-Uhlenbeck SDE;
b) α_t is found to fit into the initial discount-factors (note that by this moment we have already found x_t):

$$p^M(0,t) = \mathbb{E}^Q\left[e^{-\int_0^t (x_s + \alpha_s) ds}\right] = e^{-\int_0^t f^M(0,s)ds}$$

c) θ_t is now found to satisfy SDE $d\alpha_t = (\theta_t - a\alpha_t)dt$, that is, $\alpha_t^{'} = \theta_t - a\alpha_t$ (note that by this moment we have already found α_t).

Let's do these three steps:
a) solve Ornshtein-Uhlenbeck SDE:

$$dx_t = -ax_t dt + \sigma dW_t$$

First we apply Ito's formula to $y = f(t,x) = xe^{at}, f_t^{'} = axe^{at}, f_x^{'} = e^{at}, f_{xx}^{''} = 0$, hence obtaining the SDE that has no y_t in the rhs and so can be integrated:

$$dy_t = d(x_t e^{at}) = f_t^{'} dt + f_x^{'} dx + \frac{1}{2} f_{xx}^{''} (dx_t)^2 = ax_t e^{at} dt + e^{at} dx_t =$$

$$= ax_t e^{at} dt + e^{at}(-ax_t dt + \sigma dW_t) = \sigma e^{at} dW_t$$

$$=> y_t = y_0 + \sigma \int_0^t e^{au} dW_u = x_0 + \sigma \int_0^t e^{au} dW_u$$

$$=> \boxed{x_t = x_0 e^{-at} + \sigma \int_0^t e^{-a(t-u)} dW_u}$$

b) Find α_t: to do this we look at initial market discount-factors:

$$P^M(0,t) = \mathbb{E}^Q \left[e^{-\int_0^t (x_s + \alpha_s) ds} \right] = e^{-\int_0^t f(0,s) ds}$$

$$=> \mathbb{E}^Q \left[e^{-\int_0^t x_s ds} \right] = e^{\int_0^t \alpha_s ds - \int_0^t f(0,s) ds} \tag{5}$$

To find the expectation in the lhs, we calculate $\int_0^t x_s ds$ by changing the order of integration:

$$x_t = x_0 e^{-at} + \sigma \int_0^t e^{-a(t-u)} dW_u; x_0 = 0$$

$$=> \int_0^t x_s ds = \sigma \int_0^t \int_0^s e^{-a(s-u)} dW_u ds = \sigma \int_0^t dW_u \int_u^t e^{-a(s-u)} ds = \sigma \int_0^t e^{au} dW_u \int_u^t e^{-as} ds =$$

$$= \sigma \int_0^t e^{au} dW_u \cdot -\frac{1}{a} e^{-as}|_u^t = -\frac{\sigma}{a} \int_0^t e^{au}(e^{-at} - e^{-au}) dW_u = \frac{\sigma}{a} \int_0^t (a - e^{-a(t-u)}) dW_u$$

$$=> -\int_0^t x_s ds \sim \mathcal{N}\left(0, \frac{\sigma^2}{a^2} \int_0^t (1 - e^{-a(t-u)})^2 du\right)$$

And since if $\xi \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E} e^{\xi} = e^{\mu + \frac{\sigma^2}{2}}$, we obtain:

$$\mathbb{E}^Q \left[e^{-\int_0^t x_s ds} \right] = e^{\frac{\sigma^2}{2a^2} \int_0^t (1 - e^{-a(t-u)})^2 du} =^? e^{\int_0^t \alpha_s ds - \int_0^t f^M(0,s) ds}$$

$$=> \frac{\sigma^2}{2a^2} \int_0^t (1 - e^{-a(t-u)})^2 du = \int_0^t \alpha_s ds - \int_0^t f^M(0,s) ds$$

By differentiating both sides with respect to t, we obtain:

$$\boxed{\alpha_t} = f^M(0,t) + \frac{\sigma^2}{2a^2} \int_0^t \left[(1 - e^{-a(t-u)})^2 \right]_t^{'} du = f^M(0,t) + \frac{\sigma^2}{2a^2} \int_0^t 2(1 - e^{-a(t-u)}) a e^{-a(t-u)} du =$$

$$= f^M(0,t) - \frac{\sigma^2}{a^2}\int_0^t (1-e^{-a(t-u)}) d\left(1-e^{-a(t-u)}\right) = f^M(0,t) - \frac{\sigma^2}{2a^2}(1-e^{-a(t-u)})^2|_0^t =$$

$$= \boxed{f^M(0,t) + \frac{\sigma^2}{2a^2}(1-e^{-at})^2}$$

Finally,

$$\boxed{r_t = x_t + \alpha_t = f^M(0,t) + \frac{\sigma^2}{2a^2}(1-e^{-at})^2 + \sigma\int_0^t e^{-a(t-u)} dW_u.}$$

c) Now we find θ_t from the formula $\alpha_t^{'} = \theta_t - a\alpha_t$:

$$\boxed{\theta_t} = \alpha_t^{'} + a\alpha_t = \frac{\partial f^M(0,t)}{\partial t} + \frac{\sigma^2}{2a^2}2(1-e^{-at})ae^{-at} + a\left[f^M(0,t) + \frac{\sigma^2}{2a^2}(1-e^{-at})^2\right] =$$

$$= \frac{\partial f^M(0,t)}{\partial t} + af^M(0,t) + \frac{\sigma^2}{2a}\left[2e^{-at} - 2e^{-2at} + 1 - 2e^{-at} + e^{-2at}\right] =$$

$$= \boxed{\frac{\partial f^M(0,t)}{\partial t} + af^M(0,t) + \frac{\sigma^2}{2a}\left(1-e^{-2at}\right)}$$

Note that in case of term-structure of volatility, θ_t will take the form:

$$\boxed{\theta_t = \frac{\partial f^M(0,t)}{\partial t} + af^M(0,t) + \int_0^t \sigma_u^2 e^{-2a(t-u)} du}$$

Also not that the third term in the expression of θ_t coincides with mean-reversion parameter y_t in LGM model (but it is not intentionally, there is no hidden sence here).

4.3 HW: E and Var of $\int_0^t r_s ds$ in T-forward measure

Now from explicit formula for r_t we see that in Hull-White model r_t follows normal distribution in Q-measure, so it follows normal distribution in T_i-forward measure, and so $\int_{T_{i-1}}^{T_i} r_u du$ also follows normal distribution.

Recall that in Q-measure x_t follows SDE

$$dx_t = -ax_t dt + \sigma dW_t$$

By using Girsanov theorem, we obtain that in T-forward measure it follows SDE:

$$dx_t = -\left[\frac{\sigma^2}{a}(1-e^{-a(T-t)}) + ax_t\right] dt + \sigma dW_t^T$$

Hence we can write an explicit formula for x_t. Hence an explicit formula for

$$\int_t^T x_u du = Ax_t + B + Stoch.Term$$

Hence

$$\boxed{\mathbb{E}^T\left[\int_t^T r_u du | F_{t_0}\right] = -\int_{t_0}^T M^T(t_0,u)du + \int_{t_0}^T \alpha_u du,}$$

where

$$M^T(s,t) = \frac{\sigma^2}{a^2}\left(1 - e^{-a(t-s)}\right)\frac{\sigma^2}{2a^2}\left(e^{-a(T-t)} - e^{-a(T+t-2s)}\right)$$

And

$$\boxed{Var^T\left[\int_t^T r_u du | F_{t_0}\right] = \frac{\sigma^2}{a^2}\left(T - t + \frac{2}{a}e^{-a(T-t) - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}}\right)}$$

5 LGM model

5.1 LGM: Model setup

$$\begin{cases} dx_t = (y_t - \lambda x_t)dt + \sigma dW_t, x_0 = x_0 \\ dy_t = (\sigma_t^2 - 2\lambda y_t)dt, y_0 = 0 \\ r_t = f^M(0,t) + x_t \end{cases}$$

This model admits analytical solutions (we will derive them):

$$\begin{cases} x_t = x_0 e^{-\lambda(t-t_0)} + \int_{t_0}^t e^{-\lambda(t-u)} y_u du + \int_{t_0}^t e^{-\lambda(t-u)}\sigma_u dW_u \\ y_t = \int_0^t e^{-2\lambda(t-u)}\sigma_r(u)^2 du \\ p(t,T) = \frac{p(0,T)}{p(0,t)} e^{-G(t,T)x_t - \frac{1}{2}G(t,T)^2 y_t} \\ G(t,T) = \int_t^T e^{-\lambda(u-t)}du = \frac{1-e^{-\lambda(T-t)}}{\lambda} \end{cases}$$

We will show that:

$$\boxed{\mathbb{E}^Q\left[\int_{t_0}^t r_u du | F_{t_0}\right] = \int_{t_0}^t f_{t_0}(s)ds + \frac{1}{\lambda}\left(1 - e^{-\lambda(t-t_0)}\right)x_{t_0} + \frac{1}{2\lambda^2}\int_0^t \left(e^{-\lambda(t_0-s)^+} - e^{-\lambda(t-s)}\right)^2 \sigma_s^2 ds}$$

$$\boxed{Var^Q\left[\int_{t_0}^t r_u du | F_{t_0}\right] = \frac{1}{\lambda^2}\int_{t_0}^t \left(1 - e^{-\lambda(t-s)}\right)^2 \sigma_s^2 ds}$$

5.2 LGM: connection to Hull-White model

As derived above, in Hull-White model we have

$$r_t = x_t + \alpha_t = f^M(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2 + \sigma \int_0^t e^{-a(t-u)} dW_u.$$

In LGM model, in next subsection we will derive that

$$r_t = f^M(0,t) + x_t = f^M(0,t) + x_0 e^{-\lambda t} + \int_0^t e^{-\lambda(t-u)} y_u du + \int_0^t e^{-\lambda(t-u)} \sigma_u dW_u =$$

$$= f^M(0,t) + \frac{\sigma^2}{\lambda} \int_0^t e^{-\lambda(t-s)} \left(1 - e^{-\lambda(t-s)}\right) ds + \int_0^t e^{-\lambda(t-u)} \sigma_u dW_u$$

Obviously

$$\int_0^t e^{-\lambda(t-s)} \left(1 - e^{-\lambda(t-s)}\right) ds = \int_0^t \left(e^{\lambda(s-t)} - e^{2\lambda(s-t)}\right) ds =$$

$$= \frac{1}{\lambda} e^{\lambda(s-t)}|_0^t - \frac{1}{2\lambda} e^{2\lambda(s-t)}|_0^t = \frac{1}{\lambda}(1 - e^{-\lambda t}) - \frac{1}{2\lambda}(1 - e^{-2\lambda t}) = \frac{(1 - e^{-\lambda t})^2}{2\lambda}$$

Hence,

$$\frac{\sigma^2}{\lambda} \int_0^t e^{-\lambda(t-s)} \left(1 - e^{-\lambda(t-s)}\right) ds = \frac{\sigma^2}{2\lambda^2}(1 - e^{-\lambda t})^2$$

This means that (at least in case of constant σ), Hull-White and LGM models are identical.

5.3 LGM: connection to HJM model

A general HJM model is given in Q-measure (drift is given by HJM-drift condition)

$$df(t,T) = \sigma_f(t,T) \left(\int_t^T \sigma_f(t,s) ds \right) dt + \sigma_f(t,T) dW_t$$

The dynamics of r_t will be Markov if $\sigma_f(t,T) = h(t)g(T)$. For this case after solving SDE we obtain:

$$f(t,T) = f(0,t) + \frac{g(T)}{g(t)} \left(x_t + y_t \frac{1}{g(t)} \int_0^T g(s) ds \right),$$

where

$$\begin{cases} dx_t = \left(\frac{g'(t)}{g(t)} x_t + y_t \right) dt + g(t)h(t) dW_t \\ dy_t = \left(g^2(t)h^2(t) + 2\frac{g'(t)}{g(t)} y_t \right) dt \end{cases}$$

Let's denote

$$\begin{cases} \frac{g'(t)}{g(t)} := -\lambda(t) \\ g(t)h(t) := \sigma_r(t, x_t, y_t) \end{cases}$$

Hence we obtain LGM (and Cheyette) formulation:

$$\begin{cases} dx_t = (y_t - \lambda(t)x_t)dt + \sigma_r(t, x_t, y_t)sW_t \\ dy_t = (\sigma_r^2(t, x_t, y_t) - 2\lambda y_t)dt \end{cases}$$

5.4 HJM: reminder of HJM drift condition derivation

5.5 LGM: derivation of bond price $p(t,T)$

5.6 LGM: connection between implied and local vol

Let v be implied volatility for forward rate. The local volatility in LGM will be $\sigma_r(t)^2 = 2\mu v^2 t + v^2$. It follows from relation for total variance:

$$V(0,t) = v^2 t = \int_0^t e^{-2\mu(t-u)}\sigma_r(u)^2 du$$

$$=> v^2 t e^{2\mu t} = \int_0^t e^{2\mu u}\sigma_r(u)^2 du$$

$$=> (v^2 t e^{2\mu t})'_t = e^{2\mu t}\sigma_r(t)^2$$

$$=> v^2 e^{2\mu t} + 2\mu v^2 t e^{2\mu t} = e^{2\mu t}\sigma_r(t)^2$$

$$=> v^2 + 2\mu t v^2 = \sigma_r(t)^2$$

5.7 LGM: integration of SDE with $x(0) = 0$

a) First we solve SDE for y_t, by first solving $dy_t = -2\lambda y_t$

$$=> y_t = C(t)e^{-2\lambda t}$$

$$=> \dot{y}_t = \dot{C}e^{-2\lambda t} - 2\lambda C e^{-2\lambda t} =? \sigma_r^2 - 2\lambda y_t$$

$$=> \dot{C} = \sigma_r^2 e^{2\lambda t}$$

$$=> C(t) = \int_0^t e^{2\lambda s}\sigma_s^2 ds$$

$$=> y_t = e^{-2\lambda t}\int_0^t e^{2\lambda s}\sigma_s^2 ds = \int_0^t e^{-2\lambda(t-s)}\sigma_s^2 ds$$

b) Solve SDE for x_t:

$$dx_t = (y_t - \lambda_t x_t)dt + \sigma dW_t; x_0 = 0$$

First let's consider

$$\begin{cases} z_t = x_t e^{\lambda t} \\ z_{t_0} = x_0 = 0 \end{cases}$$

Apply Ito's formula: $f_t^{'} = \lambda x_t e^{\lambda t}, f_x^{'} = e^{\lambda t}, f_{xx}^{''} = 0$

$$=> dz_t = \lambda x_t e^{\lambda t} dt + e^{\lambda t}\left((y_t - \lambda x_t)dt + \sigma dW_t)\right) = e^{\lambda t} y_t dt + e^{\lambda t}\sigma_t dW_t$$

$$=> z_t = \int_0^t e^{\lambda s} y_s ds + \int_0^t e^{\lambda s}\sigma_s dW_s$$

$$= \int_0^t e^{\lambda u}\left(\int_0^u e^{-2\lambda(u-s)\sigma_s^2 ds}\right) du + \int_0^t e^{\lambda s}\sigma_s dW_s$$

$$= \int_0^t \int_0^u e^{\lambda(2s-u)}\sigma_s^2 ds du + \int_0^t e^{\lambda s}\sigma_s dW_s$$

$$= \int_0^t ds \left(\int_s^t du \cdot e^{\lambda(2s-u)}\right)\sigma_s^2 + \int_0^t e^{\lambda s}\sigma_s dW_s$$

$$= \int_0^t ds \left(-\frac{1}{\lambda} e^{\lambda(2s-u)}|_{u=s}^{u=t}\right)\sigma_s^2 + \int_0^t e^{\lambda s}\sigma_s dW_s$$

$$= \frac{1}{\lambda}\int_0^t \left(e^{\lambda s} - e^{\lambda(2s-t)}\right)\sigma_s^2 ds + \int_0^t e^{\lambda s}\sigma_s dW_s$$

$$=> \boxed{x_t = e^{-\lambda t} z_t = \frac{1}{\lambda}\int_0^t \left(e^{-\lambda(t-s)} - e^{-2\lambda(t-s)}\right)\sigma_s^2 ds + \int_0^t e^{-\lambda(t-s)}\sigma_s dW_s}$$

$$=> x_t \sim \mathcal{N}\left(f^M(0,t) + \frac{1}{\lambda}\int_0^t \left(e^{-\lambda(t-s)} - e^{-2\lambda(t-s)}\right)\sigma_s^2 ds, \int_0^t e^{-2\lambda(t-s)}\sigma_s ds\right)$$

5.8 LGM: integration of SDE with $x(t_0) = x_0$

a) First we solve SDE for y_t, by first solving $dy_t = -2\lambda y_t$

$$=> y_t = C(t)e^{-2\lambda t}$$

$$=> \dot{y}_t = \dot{C}e^{-2\lambda t} - 2\lambda C e^{-2\lambda t} =? \sigma_r^2 - 2\lambda y_t$$

$$=> \dot{C} = \sigma_r^2 e^{2\lambda t}$$

$$=> C(t) = \int_0^t e^{2\lambda s}\sigma_s^2 ds$$

$$=> y_t = e^{-2\lambda t}\int_0^t e^{2\lambda s}\sigma_s^2 ds = \int_0^t e^{-2\lambda(t-s)}\sigma_s^2 ds$$

b) Solve SDE for x_t:

$$dx_t = (y_t - \lambda_t x_t)dt + \sigma dW_t; x_{t_0} = x_0$$

First let's consider

$$\begin{cases} z_t = x_t e^{\lambda(t-t0)} \\ z_{t_0} = x_0 \end{cases}$$

Apply Ito's formula: $f_t' = \lambda x_t e^{\lambda(t-t_0)}, f_x' = e^{\lambda(t-t_0)}, f_{xx}'' = 0$

$$=> dz_t = z_{t_0} + \int_{t_0}^{t} e^{\lambda(u-t_0)} y_u du + \int_{t_0}^{t} e^{\lambda(u-t_0)} \sigma_u dW_u$$

$$=> \boxed{x_t = z_t e^{-\lambda(t-t_0)} = x_{t_0} e^{-\lambda(t-t_0)} + \int_{t_0}^{t} e^{\lambda(u-t)} y_u du + \int_{t_0}^{t} e^{\lambda(u-t)} \sigma_u dW_u}$$

Now we plug in the last equation the expression for y_t, found above:

$$=> x_t = x_{t_0} e^{-\lambda(t-t_0)} + \int_{t_0}^{t} e^{\lambda(u-t)} du \left(\int_0^u e^{-2\lambda(u-s)} \sigma_s^2 ds \right) + \int_{t_0}^{t} e^{\lambda(u-t)} \sigma_u dW_u =$$

$$= x_{t_0} e^{-\lambda(t-t_0)} + \int_{t_0}^{t} du \left(\int_0^u e^{\lambda(-u-t+2s)} \sigma_s^2 ds \right) + \int_{t_0}^{t} e^{\lambda(u-t)} \sigma_u dW_u =$$

$$= x_{t_0} e^{-\lambda(t-t_0)} + \int_0^{t_0} \sigma_s^2 ds \int_{t_0}^{t} e^{\lambda(-u-t+2s)} du + \int_{t_0}^{t} \sigma_s^2 ds \int_s^t e^{\lambda(-u-t+2s)} du + \int_{t_0}^{t} e^{\lambda(u-t)} \sigma_u dW_u =$$

$$= x_{t_0} e^{-\lambda(t-t_0)} + \int_0^{t_0} \sigma_s^2 ds \frac{1}{\lambda} \left[e^{\lambda(-t_0-t+2s)} - e^{\lambda(-t-t+2s)} \right] +$$

$$+ \int_{t_0}^{t} \sigma_s^2 ds \frac{1}{\lambda} \left[e^{\lambda(-s-t+2s)} - e^{\lambda(-t-t+2s)} \right] + \int_{t_0}^{t} e^{\lambda(u-t)} \sigma_u dW_u =$$

$$= x_{t_0} e^{-\lambda(t-t_0)} + \int_0^{t_0} \sigma_s^2 ds \frac{1}{\lambda} \left[e^{-\lambda(t-s+t_0-s)} - e^{-2\lambda(t-s)} \right] +$$

$$+ \int_{t_0}^{t} \sigma_s^2 ds \frac{1}{\lambda} \left[e^{-\lambda(t-s)} - e^{-2\lambda(t-s)} \right] + \int_{t_0}^{t} e^{\lambda(u-t)} \sigma_u dW_u =$$

$$= x_{t_0} e^{-\lambda(t-t_0)} - \frac{1}{\lambda} \int_0^t e^{-2\lambda(t-s)} \sigma_s^2 ds + \frac{1}{\lambda} \int_0^{t_0} \sigma_s^2 ds e^{-\lambda(t-s+t_0-s)} + \frac{1}{\lambda} \int_{t_0}^{t} \sigma_s^2 ds e^{-\lambda(t-s)} + \int_{t_0}^{t} e^{\lambda(u-t)} \sigma_u dW_u =$$

$$= x_{t_0} e^{-\lambda(t-t_0)} + \frac{1}{\lambda} \int_0^t \left(e^{-\lambda[(t-s)+(t_0-s)^+]} - e^{-2\lambda(t-s)} \right) \sigma_s^2 ds + \int_{t_0}^{t} e^{\lambda(u-t)} \sigma_u dW_u =$$

$$= \boxed{x_{t_0} e^{-\lambda(t-t_0)} + \frac{1}{\lambda} \int_0^t e^{-\lambda(t-s)} \left(e^{-\lambda(t_0-s)^+} - e^{-\lambda(t-s)} \right) \sigma_s^2 ds + \int_{t_0}^{t} e^{\lambda(u-t)} \sigma_u dW_u} =$$

$$= x_{t_0} e^{-\lambda(t-t_0)} + M(t_0, t) + \int_{t_0}^{t} e^{\lambda(u-t)} \sigma_u dW_u$$

5.9 LGM: derivation of moments of $\int_{t_0}^{t} r_u du$

So we have found expression for x_s, now we compute $\int_{t_0}^{t} x_u du$:

$$\int_{t_0}^{t} x_u du = \int_{t_0}^{t} \left(x_0 e^{-\lambda(u-t_0)} + M(t_0,u) + \int_{t_0}^{u} e^{-\lambda(u-s)} \sigma_s dW_s \right) du =$$

$$= x_0 \int_{t_0}^{t} e^{-\lambda(u-t_0)} du + \int_{t_0}^{t} M(t_0,u) du + \int_{t_0}^{t} du \int_{t_0}^{u} e^{-\lambda(u-s)} \sigma_s ds =$$

$$= \frac{1}{\lambda}(1 - e^{-\lambda(t-t_0)}) + \int_{t_0}^{t} M(t_0,u) du + \int_{t_0}^{t} \sigma_s dW_s \int_{s}^{t} e^{-\lambda(u-s)} du =$$

$$= \boxed{\frac{x_0}{\lambda}(1 - e^{-\lambda(t-t_0)}) + \int_{t_0}^{t} M(t_0,u) du + \int_{t_0}^{t} \frac{1}{\lambda}(1 - e^{-\lambda(t-s)}) \sigma_s dW_s}$$

It remains to compute $\int_{t_0}^{t} M(t_0,u) du$.

$$\int_{t_0}^{t} M(t_0,u) du = \frac{1}{\lambda} \int_{t_0}^{t} \int_{0}^{u} e^{-\lambda(u-s)} (e^{-\lambda(t_0-s)^+} - e^{-\lambda(u-s)}) \sigma_s^2 ds du =$$

$$= \frac{1}{\lambda} \int_{t_0}^{t} du \int_{0}^{t_0} e^{-\lambda(u-s)} (e^{-\lambda(t_0-s)} - e^{-\lambda(u-s)}) \sigma_s^2 ds + \frac{1}{\lambda} \int_{t_0}^{t} du \int_{t_0}^{u} e^{-\lambda(u-s)} (1 - e^{-\lambda(u-s)}) \sigma_s^2 ds =$$

$$= \frac{1}{\lambda} \int_{0}^{t_0} \sigma_s^2 ds \int_{t_0}^{t} e^{-\lambda(u-s)} (e^{-\lambda(t_0-s)} - e^{-\lambda(u-s)}) du + \frac{1}{\lambda} \int_{t_0}^{t} \sigma_s^2 ds \int_{s}^{t} e^{-\lambda(u-s)} (1 - e^{-\lambda(u-s)}) du =$$

$$= \frac{1}{\lambda} \int_{0}^{t_0} \sigma_s^2 ds \int_{t_0}^{t} (e^{\lambda(2s-t_0-u)} - e^{2\lambda(s-u)}) du + \frac{1}{\lambda} \int_{t_0}^{t} \sigma_s^2 ds \int_{s}^{t} (e^{\lambda(s-u)} - e^{2\lambda(s-u)}) du =$$

$$= \frac{1}{\lambda} \int_{0}^{t_0} \sigma_s^2 ds \frac{1}{2\lambda} (2e^{2\lambda(s-t_0)} - 2e^{\lambda(2s-t_0-t)} + e^{2\lambda(s-t)} - e^{2\lambda(s-t_0)}) + \frac{1}{\lambda} \int_{t_0}^{t} \sigma_s^2 ds \frac{1}{2\lambda} (2(1 - e^{\lambda(s-t)}) + e^{2\lambda(s-t)} - 1$$

$$= \frac{1}{2\lambda^2} \int_{0}^{t_0} \sigma_s^2 ds (e^{-2\lambda(t-s)} - 2e^{-\lambda(t-s+t_0-s)} + e^{-2\lambda(t_0-s)}) + \frac{1}{2\lambda^2} \int_{t_0}^{t} \sigma_s^2 ds (e^{-2\lambda(t-s)} - 2e^{-\lambda(t-s)} + 1) =$$

$$= \frac{1}{2\lambda^2} \int_{0}^{t_0} (e^{-\lambda(t-s)} - e^{-\lambda(t_0-s)})^2 \sigma_s^2 ds + \frac{1}{2\lambda^2} \int_{t_0}^{t} \sigma_s^2 ds (1 - - e^{-\lambda(t-s)})^2 \sigma_s^2 ds =$$

$$= \boxed{\frac{1}{2\lambda^2} \int_{0}^{t} (e^{-\lambda(t_0-s)^+} - e^{-\lambda(t-s)})^2 \sigma_s^2 ds}$$

Hence

$$\int_{t_0}^{t} x_u du | F_{t_0} = \frac{x_0}{\lambda}(1 - e^{-\lambda(t-t_0)}) + \int_{t_0}^{t} M(t_0,u) du + \int_{t_0}^{t} \frac{1}{\lambda}(1 - e^{-\lambda(t-s)}) \sigma_s dW_s =$$

$$= \boxed{\frac{x_0}{\lambda}(1 - e^{-\lambda(t-t_0)}) + \frac{1}{2\lambda^2} \int_{0}^{t} (e^{-\lambda(t_0-s)^+} - e^{-\lambda(t-s)})^2 \sigma_s^2 ds + \int_{t_0}^{t} \frac{1}{\lambda}(1 - e^{-\lambda(t-s)}) \sigma_s dW_s}$$

Hence

$$\begin{cases} \mathbb{E}^Q\left[\int_{t_0}^t x_u du|F_{t_0}\right] = \frac{x_0}{\lambda}(1-e^{-\lambda(t-t_0)}) + \frac{1}{2\lambda^2}\int_0^t (e^{-\lambda(t_0-s)^+} - e^{-\lambda(t-s)})^2\sigma_s^2 ds \\ Var^Q\left[\int_{t_0}^t x_u du|F_{t_0}\right] = \frac{1}{\lambda^2}(1-e^{-\lambda(t-s)})^2\sigma_s ds \end{cases}$$

Hence

$$\begin{cases} \mathbb{E}^Q\left[\int_{t_0}^t r_u du|F_{t_0}\right] = \int_{t_0}^t f(0,s)ds + \frac{x_0}{\lambda}(1-e^{-\lambda(t-t_0)}) + \frac{1}{2\lambda^2}\int_0^t (e^{-\lambda(t_0-s)^+} - e^{-\lambda(t-s)})^2\sigma_s^2 ds \\ Var^Q\left[\int_{t_0}^t r_u du|F_{t_0}\right] = \frac{1}{\lambda^2}(1-e^{-\lambda(t-s)})^2\sigma_s ds \end{cases}$$