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Optimal pair trading: consumption-investment problem

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Abstract We expose a simple solution of the consumption-investment problem pair trading. The proof is based on the remark that the HJB equation can be reduced to a linear parabolic equation solvable explicitly.

Keywords spread trading · pair trading · Ornstein–Uhlenbeck process · consumption-invetsment problem · HJB equation

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1 Introduction

This note contains a short proof of the recent result by Sahar Albosaily and Serguei Pergamenchtchikov [1] on the consumption-investment optimal control problem in a pair trade setting. The pair trading is based on the idea that stocks of companies with the same business are strongly correlated and their difference fluctuates near zero. A trader matches a long position with a short position in two stocks having a high correlation. The portfolio value increment is proportional to the increment of the spread between prices. By this reason such a setting, frequently used by hedge funds, is also called spread trading. The mentioned paper [1] contains an extension of the model considered earlier by Elena Boguslavskaya and Mikhail Boguslavsky in [2] where the spread was modeled by the Ornstein–Uhlenbeck process and investor's goal is to maximize only the expected utility of the terminal wealth. The HJB equation in [2], though looking rather involved, admits a solution which can be referred to

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as an explicit one. In [1] the functional to optimize includes also the expected utility of consumption with the same power utility function. The suggested analysis of the latter, rather involved and lengthy, is based on a fixed point method. It leads to the optimal solution which is the main contribution , see Ths. 5.1 and 5.3 in [1]. Though it may happen that the ideas of [1] could be useful in a more general context, they are not needed in the considered case. Here we provide arguments showing that the spread trading problem is not much more complicated than the classical Merton problem. The key ingredient of our proof is a reduction of the HJB equation to a linear parabolic equation admitting explicit solution.

2 Model

First we recall briefly the formulation of optimal control problem for spread trading. Its dynamic on $[t,T]$ is given by the two-dimensional process $(X_v, S_v)_{v\in[t,T]}$ with

$$\begin{aligned} dX_v &= (rX_v - \kappa_1 a_v S_v - c_v)dv + \alpha_v \sigma dW_v, \qquad & X_t = x,\\ dS_v &= -\kappa S_v dv + \sigma dW_v, & S_t = s. \end{aligned}$$

The constants $r \geq 0$, $\sigma, \kappa > 0$ are, respectively, the interest rate, market volatility, and mean-reverting parameter, and $\kappa_1 := r + \kappa$. The admissible control processes $\mathbf{u} := (a, c) = (a_v, c_v)_{v\in[t,T]}$ are predictable with respect to the filtration $\mathbf{F}^t$ formed by the σ-algebras $\mathcal{F}^t_v := \sigma\{W_u - W_t,\ t \leq \theta \leq v\}$ and have trajectories in $L^2([t,T]) \times L^1_+([t,T])$. Moreover, the control process vanishes after the instant when the component X attains zero. The set of such controls is denoted by $\mathcal{A}_t$. Note also that S does not depend on control, while $X = X^{\mathbf{u}}$ does.

The Bellman function of the problem is

$$J^*(x,s,t) := \sup_{\mathbf{u}\in\mathcal{A}_t} \mathbf{E}\left[\int_t^T c_v^\gamma dv + \beta X_T^\gamma\right] \tag{2.1}$$

where $0 < \gamma < 1$, $\beta > 0$. It is easily seen that the function $x \mapsto J^*(x,s,t)$ is concave and even homogeneous of order γ.

We assume, as in [1], that $\kappa \geq r$.

3 Verification lemma

The verification method is the simplest way to find the solution of optimal control problem. It prescribes to consider the Hamilton–Jacobi–Bellman (HJB) equation

$$\sup_{(a,c)\in\mathbb{R}\times\mathbb{R}_+} H(x,s,t,a,c,z(x,s,t)) = 0, \quad z(x,s,T) = \beta x^\gamma, \tag{3.1}$$

where the operator H is given by the formula

$$H(x,s,a,c,z) := z_t + (rx - \kappa_1 a - c)z_x - \kappa s z_s + \frac{1}{2}\sigma^2 z_{ss} + a\sigma^2 z_{xs} + \frac{1}{2}a^2\sigma^2 z_{xx} + c^\gamma.$$

Let $z = z(x,s,t) \geq 0$ be a classical supersolution of the HJB equation, that is a function such that $H(x,s,a,c,z) \leq 0$ for all (a,c). Then z dominates the Bellman finction J^*. Indeed, take arbitrary $\mathbf{u} \in \mathcal{A}_t$. By the Ito formula for $\theta \in [t,T]$

$$z(X_\theta^{\mathbf{u}}, S_\theta, t) + \int_t^\theta c_v^\gamma dv - z(x,s,t) = M_\theta^{\mathbf{u}} + \int_t^\theta H(X_v^{\mathbf{u}}, S_v, a_v, c_v, z(X_v^{\mathbf{u}}, S_v, t))dv,$$

where the stochastic integral

$$M_\theta^{\mathbf{u}} := \int_t^\theta \big((rX_v^{\mathbf{u}} - \kappa_1 a_v - c_v) z_x(X_v^{\mathbf{u}}, S_v, t) - \kappa S_v z_s(X_v^{\mathbf{u}}, S_v, t)\big) dW_v$$

as a process on the interval $[t,T]$ is a local martingale. It is easily seen that $M^{\mathbf{u}}$ is bounded from below, hence, it is a supermaringale and $\mathbf{E} M_\theta^{\mathbf{u}} \leq 0$ for every $\theta \in [t,T]$. It follows that

$$z(x,s,t) \geq \mathbf{E}\left[\int_t^T c_v^\gamma dv + \beta (X_T^{\mathbf{u}})^\gamma\right].$$

Since u is arbitrary, this implies that $z(x,s,t) \geq J^*(x,s,t)$.

The following assertion is usually referred to as the verification lemma (or theorem). We use the following version.

Lemma 3.1 *Let $z(x,s,t) \geq 0$ be a solution of (3.1) which is C^2 in (x,s), C^1 in t, and concave in x. Let $\mathbf{u} \in \mathcal{A}_t$ be such that the process $H(X^{\mathbf{u}}, S, a, c, z(X,S,t)) \equiv 0$ (a.s.). Define the family of random variables $\mathcal{Z} := \{z(X_\tau, S_\tau, t)\}$ where τ runs the set of stopping times with values in $[t,T]$. If $\mathcal{Z}$ is uniformly integrable, then $\mathbf{u}$ is the optimal control and $z(x,s,t) = J^*(x,s,t)$.*

4 HJB equation

Note that the supremum in a and c in the formula (3.1) is attained at

$$\tilde{a} = \frac{\kappa_1 s z_x - \sigma^2 z_{xs}}{\sigma^2 z_{xx}}, \qquad \tilde{c} = \Big(\frac{z_x}{\gamma}\Big)^{\frac{1}{\gamma-1}}. \tag{4.1}$$

Thus, the Cauchy problem (3.1) can be reduced to the Cauchy problem

$$z_t + \frac{\sigma^2}{2} z_{ss} - \frac{(\sigma^2 z_{xs} - \kappa_1 s z_x)^2}{2\sigma^2 z_{xx}} + r x z_x - \kappa s z_s + (1-\gamma)\Big(\frac{z_x}{\gamma}\Big)^{\frac{\gamma}{\gamma-1}} = 0 \tag{4.2}$$

with the terminal value $z(x,s,T) = \beta x^\gamma$.

The equation above is nonlinear but a change of variable tansforms it to a linear parabolic equation. Namely, we have

Lemma 4.1 *The solution of the terminal Cauchy problem for (4.2) admits the representation $z(x,s,t) = x^\gamma u^{1-\gamma}(s,t)$ where $u(s,t)$ is the solution of the problem*

$$u_t + \mathcal{L}u + 1 = 0, \qquad u(s,T) = \beta^{\frac{1}{1-\gamma}}, \tag{4.3}$$

where

$$\mathcal{L}u := \frac{\sigma^2}{2} u_{ss} - \big(\gamma\kappa_\gamma + \kappa\big) s u_s + \Big(\frac{1}{2\sigma^2}\gamma\kappa_\gamma^2 s^2 + r\frac{\gamma}{1-\gamma}\Big) u \tag{4.4}$$

with $\kappa_\gamma := \kappa_1/(1-\gamma)$.

Proof. The substitution $z(x,s,t)=x^\gamma y(s,t)$ in (4.2) leads to the problem

$$y_t+\frac{\sigma^2}{2}y_{ss}+\frac{\gamma}{1-\gamma}\frac{(\sigma^2 y_s/y-\kappa_1 s)^2}{2\sigma^2}y+\gamma r y-\kappa s y_s+(1-\gamma)y^{\frac{\gamma}{\gamma-1}}=0 \quad (4.5)$$

with the terminal condition $y(s,T)=\beta$.

Let $y=u^{1-\gamma}$. Substituting the formulae $y_t=(1-\gamma)u^{-\gamma}u_t$, $y_s=(1-\gamma)u^{-\gamma}u_s$, $y_{ss}=(1-\gamma)u^{-\gamma}u_{ss}-\gamma(1-\gamma)u^{-1-\gamma}u_s^2$, $y_s/y=(1-\gamma)u_s/u$ in the above equation and dividing both sides by $(1-\gamma)u^{-\gamma}$ we obtain the result. □

The representation of z given by the above lemma allows us to represent the formulae (4.1) in terms of the function $u=u(s,t)$ as follows:

$$\tilde a = xR(s,t) \text{ where } R(s,t):=\frac{u_s(s,t)}{u(s,t)}-s\frac{\kappa_\gamma}{\sigma^2},\qquad \tilde c=\frac{x}{u(s,t)}. \quad (4.6)$$

4.1 Explicit solution

Lemma 4.2 *Let g be a function satisfying the Riccati equation*

$$\dot g+\sigma^2 g^2-2\big(\gamma\kappa_\gamma+\kappa\big)g+\sigma^{-2}\gamma\kappa_\gamma^2=0 \quad (4.7)$$

and let f be a function satisfying the linear homogeneous equation

$$\dot f+\frac{\sigma^2}{2}gf+r\frac{\gamma}{1-\gamma}f=0. \quad (4.8)$$

Then the function $\tilde u(s,t):=f(t)e^{s^2 g(t)/2}$ satisfies the equation $\tilde u_t+\mathcal{L}\tilde u=0$.

Proof. We have the following expressions: $\tilde u = fe^{s^2g/2}$, $u_s=sfge^{s^2g/2}$,

$$\tilde u_t=\big(\dot f+(1/2)s^2\dot g f\big)e^{s^2g/2},\quad \tilde u_{ss}=\big(gf+s^2g^2f\big)e^{s^2g/2}.$$

Substituting them into the formula (4.4) we get the result. □

Let $g=g^\theta$ and $f=f^\theta$ be two functions satisfying on $[0,\theta]$ the equations (4.7) and (4.8) with the terminal conditions $g^\theta(\theta)=0$ and $f^\theta(\theta)=\beta^{\frac{1}{1-\gamma}}$. As an obvious corollary of the above lemma we get that the function $\tilde u^\theta(s,t):=f^\theta(t)e^{s^2g^\theta(t)/2}$ solves on $\mathbb{R}\times[0,\theta]$ the terminal Cauchy problem

$$\tilde u^\theta_t+\mathcal{L}\tilde u^\theta=0,\qquad \tilde u^\theta(s,\theta)=\beta^{\frac{1}{1-\gamma}}. \quad (4.9)$$

To alleviate formulae we skip θ when $\theta=T$.

Lemma 4.3 *The function $\tilde u(s,t):=f(t)e^{s^2g(t)/2}$ solves the problem*

$$\tilde u_t+\mathcal{L}\tilde u=0,\qquad \tilde u(s,T)=\beta^{\frac{1}{1-\gamma}}. \quad (4.10)$$

Lemma 4.4 *Let $h^\theta(s,t) := f^\theta(t)e^{s^2 g^\theta(t)/2}$. Then the function*

$$u(s,t) := \beta^{\frac{1}{\gamma-1}} \int_t^T h^\theta(s,t)d\theta + \tilde{u}(s,t) \tag{4.11}$$

solves (4.3).

Proof. Note that

$$u_t(s,t) = \beta^{\frac{1}{\gamma-1}} \int_t^T h_t^\theta(s,t)d\theta - \beta^{\frac{1}{\gamma-1}} h^t(s,t) + \tilde{u}_t(s,t).$$

and $h^t(s,t) = \beta^{\frac{1}{1-\gamma}}$. Since

$$u_t(s,t)+\mathcal{L}u(s,t)+1 = \beta^{\frac{1}{\gamma-1}} \int_t^T (h_t^\theta(s,t)+\mathcal{L}h^\theta(s,t))d\theta+\tilde{u}_t(s,t)+\mathcal{L}\tilde{u}_t(s,t) = 0,$$

the result follows from the previous lemma.

4.2 The Riccati equation with constant coefficients

Let $q = \sigma^2 g$. Then q solves the equation

$$\dot{q} + q^2 - 2(\gamma\kappa_\gamma + \kappa)q + \gamma\kappa_\gamma^2 = 0, \tag{4.12}$$

Suppose that $\kappa > r\sqrt{\gamma}$. Then the quadratic equation $\lambda^2 - 2(\gamma\kappa_\gamma + \kappa)\lambda + \gamma\kappa_\gamma^2 = 0$ with the discriminant

$$D := (\gamma\kappa_\gamma + \kappa)^2 - \gamma\kappa_\gamma^2 = \left(\gamma\frac{\kappa + r}{1-\gamma} + \kappa\right)^2 - \gamma\left(\frac{\kappa + r}{1-\gamma}\right)^2 = \frac{\kappa^2 - r^2\gamma}{1-\gamma} > 0$$

has the real roots $\lambda_1 := \gamma\kappa_\gamma + \kappa + \sqrt{D}$ and $\lambda_2 := \gamma\kappa_\gamma + \kappa - \sqrt{D} > 0$. Substituting $q = p + \lambda_2$ into (4.12) we obtain that $\dot{p} + p^2 - Ap = 0$ where $A := 2\sqrt{D} = \lambda_1 - \lambda_2$. If p does not take the zero value, then $d(1/p) = -(1/p^2)dp$. The function $P := 1/p$ satisfies the equation $\dot{P} = -AP + 1$ and can be represented as

$$P(t) = e^{-A(t-\theta)}\Big(P(\theta) + (1/A)(e^{A(t-\theta)} - 1)\Big) = (P(\theta) - (1/A))e^{-Atv-\theta)} + 1/A.$$

Since $q = \lambda_2 + 1/P$, we obtain from here an explicit formula for q. In particular, if $q(\theta) = 0$, then $P(\theta) = -1/\lambda_2$ and for $t \in [0,\theta]$

$$q^\theta(t) = \lambda_2 - \lambda_2\frac{\lambda_1 - \lambda_2}{\lambda_1 e^{(\lambda_1-\lambda_2)(\theta-t)} - \lambda_2} \geq 0, \qquad q^\theta(\theta) = 0.$$

It is easily seen that

$$\max_{0\leq t\leq\theta\leq T} q^\theta(t) = q^T(0) \leq \lambda_2 = \gamma\kappa_\gamma + \kappa - \sqrt{D}. \tag{4.13}$$

In the case where $\kappa \geq r$ the discriminant $\sqrt{D} \geq \kappa$ and, therefore,

$$g^T(0) = q^T(0)/\sigma^2 \leq \gamma\kappa_\gamma/\sigma^2. \tag{4.14}$$

4.3 Useful bounds

In the sequel C will denote a constant which value is no importance; it may be different even in a chain of formulae. To simplify formulae we skip the dependence on t where it has no importance.

According to (4.8) the function f^θ admits an explicit expression and we have that

$$\beta^{\frac{1}{1-\gamma}} \le f^\theta(t) = \beta^{\frac{1}{1-\gamma}} \exp\left\{\int_t^\theta \Big(\frac{\sigma^2}{2} g^\theta(\nu) + \frac{r\gamma}{1-\gamma}\Big) d\nu\right\} \le C.$$

It follows that

$$\beta^{\frac{1}{1-\gamma}} \le u(s,t) \le C e^{s^2 g^T(0)/2} \le C e^{s^2 \gamma\kappa_\gamma/(2\sigma^2)}. \tag{4.15}$$

4.4 Uniform integrability

Let us consider the process X following on $[t,T]$ the stochastic differential equation whose coefficients are defined in (4.6):

$$dX_v = X_v\big(r - \kappa_1 S_v R(S_v) - 1/u(S_v)\big)dv + X_v \sigma R(S_v) dW_v, \qquad X_t^* = x.$$

It can be given in more explicit way as $X_v = x\exp\{I_v + J_v\}$ where

$$I_v := \sigma \int_t^v R(S_\nu) dW_\nu, \qquad J_v := \int_t^v F(S_\nu) d\nu. \tag{4.16}$$

$$F(S) := r - \kappa_1 S R(S) - (1/2)\sigma^2 R^2(S) - 1/u(S).$$

Our aim is to find $\delta > 1$ such that $\sup_\tau \mathbf{E} z^\delta(X_\tau, S_\tau) < \infty$ where τ runs the set of all stopping times with values in $[t,T]$. Using the upper bound from (4.15) we get that

$$z^\delta(X_\tau, S_\tau) \le C X_\tau^{\gamma\delta} u^{(1-\gamma)\delta}(S_\tau) \le C X_\tau^{\gamma\delta} e^{S_\tau^2 \delta\gamma\kappa_1/(2\sigma^2)}$$

By the Ito formula applied to the square Ornstein–Uhlenbeck process

$$S_\tau^2 = s^2 + \int_t^\tau (-2\kappa S_\nu^2 + \sigma^2) d\nu + 2\sigma \int_t^\tau S_\nu dW_\nu, \quad v \in [t,T].$$

Let $p > 1$ and let p' be its conjugate, i.e. $p' := p/(p-1)$. We prepare these numbers to use the Hölder inequality to isolate a stochastic exponential of a local martingale. With such a provision we rewrite the right-hand side of the above inequality as

$$z^\delta(X_\tau, S_\tau) \le C \exp\left\{\int_t^\tau G_\nu dW_\nu - \frac{1}{2} p \int_t^\tau G_\nu^2 d\nu\right\} \exp\left\{\int_t^\tau \tilde{G}_\nu d\nu\right\} \tag{4.17}$$

where we include in C results of integration of bounded terms,

$$\begin{aligned} G &:= \delta\gamma\sigma R + (\delta\gamma\kappa_1/\sigma) S, \\ \tilde{G} &:= -\gamma\delta\kappa_1 S R - (1/2)\delta\gamma\sigma^2 R^2 - (1/\sigma^2)\kappa\delta\gamma\kappa_1 S^2 \\ &\quad + (1/2)p(\delta\gamma)^2\sigma^2 R^2 + p\delta^2\gamma^2\kappa_1 R S + (1/2)p(\delta\gamma\kappa_1/\sigma)^2 S^2 \end{aligned}$$

with the abbreviation $R := R(S)$. If we take $p\delta\gamma = 1$, then the coefficients at R^2 and RS vanish and

$$\tilde{G} = \frac{\delta\gamma\kappa_1}{2\sigma^2}(\kappa_1 - 2\kappa)S^2 = \frac{\delta\gamma\kappa_1}{2\sigma^2}(r - \kappa)S^2 \leq 0.$$

Thus, the L^p-norm of the first exponential in the rhs of (4.17) and the $L^{p'}$-norm of the second one are less or equal to one, $\mathbf{E}\big[z^{\delta p}(X_\tau, S_\tau)\big] \leq C$ and we get the needed uniform integrability property.

5 Conclusion

The consumption-investment problem in the setting of pair trade admits an explicit solution. The arguments are based on the observation that the HJB equation in the Ornstein–Uhlenbeck spread model is reduced to a linear parabolic PDE admitting an explicit solution. This observation drastically simplifies the arguments in [1].

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