# Stochastic control theory in portfolio selection

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**Abstract**

This research project investigates time risk preferences and goal-reaching problems in the context of optimal investment. Framing investment objectives as goal-reaching problems is increasingly popular in investment practice, especially in the emerging industry of Fintech. These problems lead to interesting and challenging associated stochastic control problems that are typically solved by one of two methods: Hamilton-Jacobi-Bellman equations or Pontryagin's maximum principle. The project aims to build the associated theory for both approaches and translate developed theory to practice through application of relevant stochastic control techniques to two real-world problems: famous Merton problem and the problem of beating a target. New results, obtained in this work, are the explicit solution of Merton problem by Pontryagin's maximum principle approach and explicit formula for optimal strategy in the problem of beating a constant target in the model with risk-free rate, which is the extension of existing result in the model without risk-free rate.

# Contents

# 1 Introduction

Goals-based portfolio management is an investment paradigm centered around the fulfillment of client's consumption goals. Client wants to achieve in (maybe) predefined time/times one or several predefined goals, among which may be purchasing a car or an apartment. The importance of goal-based investing has been recognized both by the academic community and by the private sector. In year 2020 Q-Group panel discussion, Robert Merton emphasized goal-based investing as one of the most important problems in financial engineering for the next decade. Merton was first to address the problem: in Merton (1969) he studied what fraction of wealth should the agent invest into the risky asset if his goal is to maximize expected utility of terminal wealth on a finite horizon or to maximize expected integrated utility of running consumption on an infinite horizon.

The workhorse method for solving stochastic optimal control problems is to find the Hamilton-Jacobi-Bellman equations for the value function and then try to solve these in some way. The second principal and most commonly used approach for solving stochastic optimal control problems is Pontryagin's stochastic maximum principle. An interesting phenomenon one can observe from the literature is that these two approaches have been developed separately and independently. Both methods are used to investigate the same problems, and connections between these two approaches can be seen in classical mechanics and are discussed in detail in Yong and Zhou (1999).

This project is focused on two connected themes in the area of portfolio management. The first goal is to explore the connections between HJB equations approach and Pontryagin's stochastic maximum principle approach and solve the famous Merton problem using seconds approach (while in the literature it is always solved using first approach, see, for example, Chapter 3 of Pham (2009)). We will consider Merton problem without consumption and with consumption.

The second goal is to study portfolio selection problems with both time risk preference and utility of wealth, using stochastic control theory from dynamic programming and HJB equations. In particular, in this project we extend the recent model from Chapter 4 of Wang (2024) for finding the optimal investment portfolio to beat a constant target by introducing the risk-free rate into the market and presenting the explicit formula for optimal strategy. This result is of particular interest to the insurance industry, since insurance companies and pension funds in their everyday work follow a benchmarking procedure, for example by trying to beat inflation, exchange rates, or other indices.

The rest of this thesis is organized as follows.

In Chapter 2 we focus on stochastic optimal control problems and aim to explore the connection between two existing approaches of solving these problems: via Pontryagin's maximum principle and via Hamilton-Jacobi-Bellman equation. We also give formal justification of stochastic Pontryagin's maximum principle together with dynamical programming principle, Hamilton-Jacobi-Bellman equation and verification theorem.

In Chapter 3 we apply the machinery of two approaches from Chapter 2 to solve the famous Merton problem. The novelty of this work is the solution of this problem via Pontryagin's maximum principle instead of via Hamilton-Jacobi-Bellman equation.

In Chapter 4 we give a brief introduction to beating a moving target problem and develop mathematical apparatus for the general problem with time risk preferences. Then we present our second main result: we explicitly find the optimal investment strategy in a problem with risk-free rate.

In Chapter 5 we conclude and discuss directions for further research.

## 2 Stochastic optimal control problems: two classical approaches

In this chapter, we use the dynamic programming method for solving stochastic control problems. We consider the framework of controlled diffusion and formulate the problem on finite or infinite horizon. The basic idea of the approach, called the dynamic programming principle, is to consider a family of control problems by varying the initial state values, and to derive some relations between the associated value functions. This approach yields a certain partial differential equation (PDE), of second order and nonlinear, called Hamilton-Jacobi-Bellman (HJB). When this PDE can be solved by the explicit or theoretical achievement of a smooth solution, the verification theorem validates the optimality of the candidate solution to the HJB equation. This classical approach to the dynamic programming is called the verification step. More details about this approach can be found in Chapter 3 of Pham (2009).

### 2.1 Controlled diffusion processes

We consider a control model where the state of the system is governed by a stochastic differential equation (SDE) with values in $\mathbb{R}^n$:

$$dX_s = b(X_s, \alpha_s)ds + \sigma_s(X_s, \alpha_s)dW_s, \qquad s \in [0, T] \tag{1}$$

where $W$ is a d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{P})$. Controlled parameter (or control) $\alpha = (\alpha_s) \subseteq A \subseteq \mathbb{R}^m$ is a progressively measurable (with respect to $\mathbb{F}$) process (not necessarily continuous), valued in $A$, subset of $\mathbb{R}^m$.
Coefficients $b : \mathbb{R}^n \times A \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times A \to \mathbb{R}^{n\times d}$ are measurable and satisfy a uniform Lipshitz condition in $A$, which means that there exists constant $K \geq 0$ such that

$$\forall x, y > 0 \in \mathbb{R}^n, \forall a \in A : |b(x, a) - b(y, a)| + |\sigma(x, a) - \sigma(y, a)| \leq K|x - y|. \tag{2}$$

We denote by $\tau_{t,T}$ the set of stopping times with values in $[t, T]$.
We will consider separately finite horizon problem and infinite horizon problem, because we will use finite horizon for Merton problem and infinite horizon for target-beating problem.

### 2.2 Finite horizon problem setting

We fix a finite time horizon $T \in (0, +\infty)$ and denote by $\mathcal{A}$ the set of control processes

$$\mathcal{A} = \left\{ \alpha \in \mathbb{R}^n : \mathbb{E}\left( \int_0^T |b(0, \alpha_t)|^2 + |\sigma(0, \alpha_t)^2|dt \right) < \infty. \right\} \tag{3}$$

According to Section 1.3 in Chapter 1 of Pham (2009), Lipshitz condition (2) and condition (3) in the definition of $\mathcal{A}$ guarantee that for all initial conditions $(t, x) \in [0, T]\times\mathbb{R}^n$ there exists a unique strong solution to the SDE (1) starting from point $x$ at time $s = t$.

Indeed, the conditions of Ito's theorem require that there exist a constant $K \in \mathbb{R}$ and a process $\kappa_t$ (a natural choice is $\kappa_t = |b(t, 0)| + |\sigma(t, 0)|$) that for all $t \in T, \omega \in \Omega, x, y \in \mathbb{R}$ :

$$\begin{cases} |(b(t, x, \omega) - b(t, y, \omega)| + |\sigma(t, x, \omega) - \sigma(t, y, \omega)| \leq K|x - y|, \\ |(b(t, x, \omega)| + |\sigma(t, x, \omega)| \leq \kappa_t(\omega) + K|x| \qquad \text{with } \mathbb{E}\int_0^t |\kappa_u|^2 du < \infty, \forall t \in T. \end{cases}$$

Let $\left\{ X_s^{t,x}; t \leq s \leq T \right\}$ be the (unique) solution of SDE with a.s continuous paths. We also recall that under these conditions on $b, \sigma, \alpha$ we have (see Theorem 1.3.16 in Pham

(2009)):

$$\begin{cases} E\left[\sup_{t\le s\le T}|X_s^{t,x}|^2\right] < \infty, \\ \lim_{h\to 0^+} E\left[\sup_{t\le s\le h}|X_s^{t,x}-x|^2\right] = 0. \end{cases} \tag{4}$$

Now we are ready to introduce the gain functional.
Let $f : [0,T]\times\mathbb{R}^n\times A \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be two measurable functions. We suppose that one of the following holds (Hg):

$$\left[\begin{array}{l}(i)\text{g is lower-bounded} \\ (ii)\text{g satisfies a quadratic growth conditions: } |g(x)| \le C(1+|x|^2), \forall x\in\mathbb{R}^n\end{array}\right.$$

For $(t,x)\in[0,T]\times\mathbb{R}^n$ we denote by $\mathcal{A}(t,x)$ the following set of controls (and we assume that $\mathcal{A}(t,x)$ is not empty for all $(t,x)\in[0,T]\times\mathbb{R}^n$):

$$\mathcal{A}(t,x) = \left\{\alpha\in\mathcal{A} : \mathbb{E}\left[\int_t^T |f(s,X_s^{t,x},\alpha_s|ds\right] < \infty\right\}.$$

We can then define under condition (Hg) the gain functional:

$$J(t,x,\alpha) = \mathbb{E}\left(\int_t^T f(s,X_s^{t,x},\alpha_s)ds + g(X_T^{t,x})\right).$$

The objective is to maximize over control processes the gain functional $J$, and we introduce the associated value function:

$$v(t,x) = \sup_{\alpha\in\mathcal{A}(t,x)} J(t,x,\alpha).$$

Given an initial condition $(t,x)\in[0,T]\times\mathbb{R}^n$, we call $\hat{\alpha}\in\mathcal{A}(t,x)$ the optimal control if $v(t,x) = J(t,x,\hat{\alpha})$. We call the control Markovian, if $\alpha_s = a(s,X_s^{t,x})$ for some measurable function $a$ from $[0,T]\times\mathbb{R}^n$ into $A$.

**Remark 1.** *a) Under (Hg) we assume that $g$ satisfies quadratic growth conditions. If we assume that $f$ also satisfies quadratic growth conditions, i.e. there exists a positive constant $C$ and a positive function $\kappa : A\to\mathbb{R}^+$, such that $|f(t,x,a)| \le C(1+|x|^2)+\kappa(a)$, then due to first part of (4) we can conclude that the constant controls from $\mathcal{A}$ lie in $\mathcal{A}(t,x)$.*

*b) Moreover, if in addition to $|f(t,x,a)| \le C(1+|x|^2)+\kappa(a)$ there exists a positive constant $C$ such that $\kappa(a) \le C(1+|b(0,a)|^2+|\sigma(0,a)|^2)$, for all $a\in A$, then conditions (3) and (4) show that for all $(t,x)\in[0,T]\times\mathbb{R}^n$, $\alpha\in\mathcal{A}$ we have: $\mathbb{E}\left[\int_t^T |f(s,X_s^{t,x},\alpha_s|ds\right] < \infty$, which means that $\mathcal{A}$ and $\mathcal{A}(t,x)$ coincide.*

### 2.3 Infinite time horizon problem setting

We denote by $\mathcal{A}$ the set of control processes $\alpha$ such that

$$\mathcal{A} = \left\{\alpha\in\mathbb{R}^n : \mathbb{E}\left(\int_0^{+\infty} |b(0,\alpha_t)|^2 + |\sigma(0,\alpha_t)^2|dt\right) < \infty.\right\} \tag{5}$$

Given an initial condition $t = 0, x \in \mathbb{R}^n$, and a control $\alpha \in \mathcal{A}$, there exists a unique strong solution of SDE (1) starting from $x$ at $t = 0$, that we denote by $\{X_s^x; ts \geq 0\}$.

According to Theorem 1.3.16 from Pham (2009), we have the following estimate:

$$\mathbb{E}\left[|X_s^x|^2\right] \leq C|x|^2 + Ce^{Cs}\mathbb{E}\left[\int_0^s |x|^2 + |b(0,\alpha_u|^2 + |\sigma(0,\alpha_u)|^2 du\right] \tag{6}$$

for some constant $C$, independent of $s, x, \alpha$.

Now we are ready to define the gain functional.

Let $\beta > 0$ and $f : \mathbb{R}^n \times A \to \mathbb{R}$ be a measurable function. We introduce set of controls $\mathcal{A}(x)$ and assume that it is not empty for all $x \in \mathbb{R}^n$.

$$\mathcal{A}(x) = \left\{\alpha \in \mathcal{A}_0 : \mathbb{E}\left(\int_0^{+\infty} e^{-\beta s}|f(X_s^x, \alpha_s)|ds\right) < \infty.\right\} \tag{7}$$

We then define the gain functional:

$$J(x, \alpha) = \mathbb{E}\left(\int_t^{+\infty} e^{-\beta s} f(X_s^x, \alpha_s)ds\right).$$

The objective is to maximize over control processes the gain functional $J$, and we introduce the associated value function:

$$v(x) = \sup_{\alpha \in \mathcal{A}(x)} J(x, \alpha).$$

Given an initial condition $x \in \mathbb{R}^n$, we call $\hat{\alpha} \in \mathcal{A}(x)$ the optimal control if $v(x) = J(x, \hat{\alpha})$. We call the control Markovian, if $\alpha_s = a(X_s^x)$ for some measurable function $a$ from $\mathbb{R}^+ \times \mathbb{R}^n$ into $A$.

Notice that it is important to suppose here that the function $f(x, a)$ does not depend on time in order to get the stationarity of the problem, i.e. the value function does not depend on the initial date at which the optimization problem is considered.

**Remark 2.** *When $f$ satisfies a quadratic growth condition in $x$, i.e. there exist a positive constant $C$ and a positive function $\kappa : A \to \mathbb{R}_+$ such that $|f(x,a) \leq C(1 + |x|^2) + \kappa(a), \forall (x,a) \in \mathbb{R}^n \times A$, then estimate (6) shows that for $\beta > 0$ large enough, for all $x \in \mathbb{R}^n, a \in A$ we have $\mathbb{E}\left[\int_0^{+\infty} e^{-\beta s}|f(X_s^x, \alpha_s)|ds\right] < \infty$, meaning that the constant controls from $\mathcal{A}$ lie in $\mathcal{A}(x)$.*

## 2.4 Approach 1: dynamical programming principle, HJB equation, verification theorem

### 2.4.1 Dynamical programming principle

The dynamic programming principle (DPP) is a fundamental principle in the theory of stochastic control. In the context of controlled diffusion processes described in the previous section, and in fact more generally for controlled Markov processes, it is formulated as follows:

**Theorem 1.** *(Dynamical programming principle, Theorem 3.1.1 in Pham (2009))*

*(1) Finite horizon: let $(t, x) \in [0, T] \times \mathbb{R}^n$.*
*Then we have*

$$v(t,x) = \sup_{\alpha \in \mathcal{A}(t,x)} \sup_{\theta \in \tau_{t,T}} \mathbb{E}\left(\int_t^\theta f(s, X_s^{t,x}, \alpha_s)ds + v(\theta, X_\theta^{t,x})\right)$$

$$= \sup_{\alpha \in \mathcal{A}(t,x)} \inf_{\theta \in \tau_{t,T}} \mathbb{E} \left( \int_t^\theta f(s, X_s^{t,x}, \alpha_s) ds + v(\theta, X_\theta^{t,x}) \right).$$

*(1) Infinite horizon: let $x \in \mathbb{R}^n$.*
*Then we have*

$$v(x) = \sup_{\alpha \in \mathcal{A}(x)} \sup_{\theta \in \tau} \mathbb{E} \left( \int_t^\theta e^{-\beta s} f(X_s^x, \alpha_s) ds + e^{-\beta\theta} v(X_\theta^x) \right)$$

$$= \sup_{\alpha \in \mathcal{A}(x)} \inf_{\theta \in \tau} \mathbb{E} \left( \int_t^\theta e^{-\beta s} f(X_s^x, \alpha_s) ds + e^{-\beta\theta} v(X_\theta^x) \right).$$

*Here we denote by $\tau_{t,T}$ the set of stopping times with values in $[t, T]$.*

**Remark 3.** *In case of finite horizon :*
*a) The given above formulation of DPP is equivalent to:*

$$\begin{cases} \forall \alpha \in \mathcal{A}(t,x), \theta \in \tau_{t,T} : v(t,x) \geq \mathbb{E} \left( \int_t^\theta f(s, X_s^{t,x}, \alpha_s) ds + v(\theta, X_\theta^{t,x}) \right) \\ \forall \epsilon > 0, \exists \alpha \in \mathcal{A}(t,x) : \forall \theta \in \tau_{t,T} : v(t,x) - \epsilon \leq \mathbb{E} \left( \int_t^\theta f(s, X_s^{t,x}, \alpha_s) ds + v(\theta, X_\theta^{t,x}) \right) \end{cases}$$

*b) Also note that the given above formulation of the DPP is the stronger version of usual finite horizon dynamical programming principle (where $\theta$ is fixed):*

$$v(t,x) = \sup_{\alpha \in \mathcal{A}(t,x)} \mathbb{E} \left( \int_t^\theta f(s, X_s^{t,x}, \alpha_s) ds + v(\theta, X_\theta^{t,x}) \right), \forall \theta \in \tau_{t,T}. \tag{8}$$

The proof of this theorem can be seen in Chapter 3.3 of Pham (2009).

### 2.4.2 HJB equation derivation

The Hamilton-Jacobi-Bellman equation (HJB) is the infinitesimal version of the dynamic programming principle: it describes the local behavior of the value function when we send the stopping time $\theta$ in (8) to $t$.

For finite horizon, we put at DDP (8) $\theta := t + h$, assume that $v$ is sufficiently smooth, apply Ito's formula, go to limit $h \to 0$ and obtain that $v$ should satisfy the following equation:

$$-\frac{\partial v}{\partial t}(t,x) - \sup_{a \in A} [\mathcal{L}^a v(t,x) + f(t,x,a)] = 0, \forall (t,x) \in [0,T] \times \mathbb{R}^n.$$

We often rewrite this PDE in the form:

$$-\frac{\partial v}{\partial t}(t,x) - H(t,x,D_x v(t,x), D_{xx}^2 v(t,x)) = 0, \forall (t,x) in [0,T] \times \mathbb{R}^n \tag{9}$$

where for $(t,x,p,M) \in [0,T] \times \mathbb{R}^n \mathbb{R}^\ltimes \times S_n$

$$H(t,x,p,M) = \sup_{a \in A} \left[ b(x,a)p + \frac{1}{2} tr(\sigma\sigma^T(x,a)M) + f(t,x,a) \right].$$

This function $H$ is called the Hamiltonian of the associated control problem. The equation (10) is called the dynamic programing equation or Hamilton-Jacobi-Bellman (HJB) equation. The regular terminal condition associated to this PDE is

$$v(T, x) = g(x), \forall x \in \mathbb{R}^n.$$

For infinite horizon, we can use similar arguments as in the finite horizon case to derive formally the HJB equation for the value function:

$$\beta v(x) - \sup_{a \in A} [\mathcal{L}^a v(x) + f(x, a)] = 0, \forall x \in \mathbb{R}^n,$$

which may be rewritten also as

$$\beta v(x) - H(x, D_x v(x), D^2_{xx} v(x)) = 0, \forall x \in \mathbb{R}^n, \tag{10}$$

where for $(x, p, M) \in \mathbb{R}^n \times \mathbb{R}^n \times S_n$:

$$H(x, p, M) = \sup_{a \in A} \left[ b(x, a)p + \frac{1}{2} tr(\sigma\sigma^T(x, a)M) + f(x, a) \right].$$

### 2.4.3 Verification theorem

**Theorem 2.** *(Verification theorem for finite horizon, Theorem 3.5.2 in Pham (2009))*
*Let $\omega$ be a function in $C^{1,2}([0, T) \times \mathbb{R}^n) \cap C^0([0, T] \times \mathbb{R}^n)$ and satisfying quadratic growth condition, i.e. there exists a constant $C$ such that $|w(t, x)| \leq C(1 + |x|^2)$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$.*
*1) Suppose that*

$$\begin{cases} -\frac{\partial w}{\partial t}(t, x) - \sup_{a \in A} [\mathcal{L}^a w(t, x) + f(t, x, a)] \geq 0, (t, x) \in [0, T] \times \mathbb{R}^n, \\ w(T, x) \geq g(x), x \in \mathbb{R}^n \end{cases}$$

*Then $w \geq v$ on $[0, T] \times \mathbb{R}^n$.*

*2) Suppose further that $w(T_.) = g$, and there exists measurable control $\hat{\alpha}(t, x) \in A$ such that*

$$-\frac{\partial w}{\partial t}(t, x) - \sup_{a \in A} [\mathcal{L}^a w(t, x) + f(t, x, a)] = -\frac{\partial w}{\partial t}(t, x) - \mathcal{L}^{\hat{\alpha}(t,x)} w(t, x) - f(t, x, \hat{\alpha}(t, x)) = 0$$

*and SDE*

$$dX_s = b(X_s, \hat{\alpha}(s, X_s))ds + \sigma(X_s, \hat{\alpha}(s, X_s))dW_s$$

*admits a unique solution, denoted by $\hat{X}^{t,x}_s$, given an initial condition $X_t = x$, and the process $\left\{ \hat{\alpha}(s, \hat{X}^{t,x}_S) \right\}$ lies in $\mathcal{A}(t, x)$.*
*Then $w = v$ on $[0, T] \times \mathbb{R}^n$, and $\hat{\alpha}$ is an optimal markovian control.*

**Theorem 3.** *(Verification theorem for infinite horizon, Theorem 3.5.3 in Pham (2009))*
*Let $\omega \in C^2(\mathbb{R}^n)$ and satisfies a quadratic growth condition, i.e. there exists a constant $C$ such that $|w(x)| \leq C(1 + |x|^2)$ for all $x \in \mathbb{R}^n$.*
*1) Suppose that*

$$\begin{cases} \beta w(x) - \sup_{a \in A} [\mathcal{L}^a w(x) + f(x, a)] \geq 0, x \in \mathbb{R}^n, \\ \limsup_{T \to +\infty} e^{-\beta T} \mathbb{E} w(X^x_T), \forall x \in \mathbb{R}^n, \forall \alpha \in \mathcal{A}(x). \end{cases}$$

*Then $w \geq v$ on $\mathbb{R}^n$.*

*2) Suppose further that for all $x \in \mathbb{R}^n$ there exists measurable control $\hat{\alpha}(x) \in A$ such that*

$$\beta w(x) - \sup_{a \in A} [\mathcal{L}^a w(x) + f(x,a)] = \beta w - \mathcal{L}^{\hat{\alpha}(x)} w(x) - f(x, \hat{\alpha}(t,x)) = 0$$

*and SDE*

$$dX_s = b(X_s, \hat{\alpha}(s, X_s))ds + \sigma(X_s, \hat{\alpha}(s, X_s))dW_s$$

*admits a unique solution, denoted by $\hat{X}_s^x$, given an initial condition $X_0 = x$, satisfying $\liminf_{T \to +\infty} e^{-\beta T} E w(\hat{X}_T^x) \leq 0$, and the process $\left\{ \hat{\alpha}(s, \hat{X}_S^x) \right\}$ lies in $\mathcal{A}(x)$.*
*Then $w(x) = v(x)$ on $\mathbb{R}^n$, and $\hat{\alpha}$ is an optimal Markovian control.*

### 2.5 Approach 2: Pontryagin's stochastic maximum principle

In the previous section, we studied how to solve a stochastic control problem by the dynamic programming method. We present here an alternative approach, called Pontryagin maximum principle, that is based on optimality conditions for controls.

We consider the framework of a stochastic control problem on a finite horizon as defined before in Section 2.2.

We define the generalized Hamiltonian $\mathcal{H} : [0,T] \times \mathbb{R}^n \times A \times \mathbb{R}^{n \times d} \to \mathbb{R}$ by

$$\mathcal{H}(t,x,a,y,z) = b(x,a)y + tr(\sigma^T(x,a)z) + f(t,x,a),$$

and we assume that $\mathcal{H}$ is differentiable in $x$ with derivative denoted by $D_x\mathcal{H}$. We consider for each $\alpha \in \mathcal{A}$, the BSDE, called the adjoint equation:

$$-dY_t = D_x\mathcal{H}(t, X_t, \alpha_t, Y_t, Z_t)dt - Z_t dW_t, Y_T = D_x g(X_T). \tag{11}$$

**Theorem 4.** *(Pontryagin's stochastic maximum principle, see Theorem 6.4.6 in Pham (2009))*

*Let $\hat{\alpha} \in \mathcal{A}$ and $\hat{X}$ be the associated controlled diffusion. Suppose that there exists a solution $(\hat{Y}, \hat{Z})$ to the associated BSDE (11) such that*

$$\mathcal{H}(t, \hat{X}_t, \hat{\alpha}_t, \hat{Y}_t, \hat{Z}_t) = \max_{a \in A} \mathcal{H}(t, \hat{X}_t, a, \hat{Y}_t, \hat{Z}_t), 0 \leq t \leq T, a.s.$$

*and*

$$(x,a) \to \mathcal{H}(t,x,a,\hat{Y}_t,\hat{Z}_t) \qquad \text{is a concave function for all } t \in [0,T].$$

*Then $\hat{\alpha}$ is an optimal control, i.e.*

$$J(\hat{\alpha}) = \sup_{\alpha \in \mathcal{A}} J(t,x,\alpha).$$

We want to provide the connection between maximum principle and dynamic programming. The value function of the stochastic control problem considered above is defined by

$$v(t,x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left( \int_t^T f(s, X_s^{t,x}, \alpha_s)ds + g(X_T^{t,x}) \right), \tag{12}$$

where $\left\{ X_s^{t,x}, t \leq s \leq T \right\}$ is the solution to SDE (??), starting from $x$ at $t$. Recall that the associated Hamilton-Jacobi-Bellman equation is

$$-\frac{\partial v}{\partial t}(t,x) - \sup_{a \in A} \mathcal{G}(t,x,a,D_x v, D_{xx}^2 v) = 0, \forall (t,x) \in [0,T] \times \mathbb{R}^n, \tag{13}$$

where for $(t, x, a, p, M) \in [0, T] \times \mathbb{R}^n \times A \times \mathbb{R}^\ltimes \times S_n$ :

$$\mathcal{G}(t, x, p, M) = [b(x, a)p + \frac{1}{2}tr(\sigma\sigma^T(x, a)M) + f(t, x, a)].$$

**Theorem 5.** *(Connection between DPP and Pontryagin's maximum principle, see Theorem 6.4.7 in Pham (2009))*
*Suppose that* $v \in C^{1,3}([0, T) \times \mathbb{R}^n) \cap C^0([0, T] \times \mathbb{R}^n)$ *and there exists an optimal control* $\hat{\alpha} \in \mathcal{A}$ *to optimal control problem* (12) *with associated controlled diffusion* $\hat{X}$. *Then*

$$\mathcal{G}(t, \hat{X}_t, \hat{\alpha}t, D_x v(t, \hat{X}_t), D^2_{xx} v(t, \hat{X}_t)) = \max_{a \in A} \mathcal{G}(t, \hat{X}_t, a, D_x v(t, \hat{X}_t), D^2_{xx} v(t, \hat{X}_t))$$

*and the pair*

$$(\hat{Y}_t, \hat{Z}_t) = (D_x v(t, \hat{X}_t), D^2_{xx} v(t, \hat{X}_t)\sigma(\hat{X}_t, \hat{\alpha}_t)),$$

*is solution to the adjoint BSDE* (11).

### 2.6 Discussion

In the statement of a Pontryagin-type maximum principle there is an adjoint equation, which is an ordinary differential equation (ODE) in the (finite-dimensional) deterministic case and a stochastic differential equation (SDE) in the stochastic case. The system consisting of the adjoint equation, the original state equation, and the maximum condition is referred to as an (extended) Hamiltonian system. On the other hand, in Bellman's dynamic programming, there is a partial differential equation (PDE), of first order in the (finite-dimensional) deterministic case and of second order in the stochastic case. This is known as a Hamilton-Jacobi-Bellman (HJB) equation.

Note that one easily sees a strong analogy between optimal control and analytic mechanics. This is not surprising, however, since the classical calculus of variations, which is the foundation of analytic mechanics, is indeed the origin of optimal control theory.

For more detailed comparison of DPP and Pontryagin's maximum principle one can refer to Yong and Zhou (1999).

## 3 Application 1: Merton problem

In this section we will illustrate how to solve the well-known Merton problem with two previously defined approaches: HJB equation and Pontryagin's stochastic maximum principle. As far as we know, in the literature this problem is always solved via HJB approach, and never via Pontryagin's stochastic maximum principle.

We consider a financial market consisting of a riskless asset with strictly positive price process $B_t$ representing the savings account, and a risky asset of price process $S_t$, representing stock. An agent may invest in this market at any time $t$, with a number of shares $\alpha_t$ in the risky asset.

This process $\alpha_t$ valued in $A$, subset of $\mathbb{R}^n$, is the control. The portfolio allocation problem is to choose optimally the proportions of capital to invest in each of two assets. One of the classical modeling for describing the behavior and preferences of agents and investors is the expected utility criterion (while second well-known approach is mean-variance criterion). For a longer discussion on the preferences representation of agents, one can refer to Föllmer and Schied Föllmer and Schied (2002). In this portfolio allocation context, the criterion consists of maximizing the expected utility of terminal wealth on a finite horizon $T < +\infty$: that is, finding $\sup_\alpha u(X_T)$.

The problem originally studied by Merton in Merton (1969) is a particular case of the model described above with a Black-Scholes model for the risky asset and a power utility function.

## 3.1 Problem without consumption

We have a risky asset $S_t$ and a bank account $B_t$, with $\mu > r$, since otherwise there is no sense to invest in stock at all:

$$\begin{cases} \frac{dS_t}{S_t} = \mu dt + \sigma dW_t \\ \frac{dB_t}{B_t} = rdt \end{cases}$$

The space of controls is $U = \mathbb{R}$. We allow only for square-integrable controls since we want the solution of SDE to exist:

$$\mathcal{A} = \left\{ \alpha : [0, T] \to U : \mathbb{E} \int_0^T |\alpha_t|^2 dt < \infty \right\}.$$

Denote by $X_t$ the capital of the investor at time $t$. Investor allocates some fraction $\alpha_t$ of his capital $X_t$ into the risky asset $S_t$ (so the amount of shares of risky asset that he bought is $\frac{\alpha_t X_t}{S_t}$) and the rest $(1 - \alpha_t)X_t$ amount of money he puts onto the bank account. Hence the evolution of the investor's capital $X_t$ is described by the following SDE:

$$dX_t = \frac{\alpha_t X_t}{S_t} dS_t + \frac{(1 - \alpha_t)X_t}{B_t} dB_t.$$

Plugging here the SDE for $S_t$, we obtain:

$$dX_t = X_t\alpha_t(\mu dt + \sigma dW_t) + X_t(1 - \alpha_t)rdt = (\alpha_t\mu + (1 - \alpha_t)r)X_t dt + \alpha_t\sigma X_t dW_t$$

The investor's goal is to maximize the PnL at the last point:

$$J(\alpha) = \mathbb{E}u(X_T)$$

Utility function $u(x)$ here (and in very many other problems) is $u(x) = \frac{x^p}{p}, p \in (0, 1)$. Note that Merton problem is also solvable for another utility function: $u(x) = \ln(x)$. And it is not very surprising, since $\ln(x)$ is the limit of $\frac{x^p}{p}$ as $p \to 0$.

The value function that we want to find is

$$v(t, x) = \sup_{\alpha\in\mathcal{A}} \mathbb{E}u(X_T).$$

### 3.1.1 Solution by HJB method

The goal functional in Merton problem is the special case of the functional $J$ that we studied in Chapter 2. So we can write the HJB equation with boundary condition for $J$:

$$\begin{cases} v'_t(t, x) + \min_{\alpha\in U} \left\{ f(t, x, \alpha) + \nabla v(x, t)b(t, x, \alpha_t) + tr(\frac{1}{2}\sigma\sigma^T(t, x, \alpha_t)D^2v(t, x)) \right\} = 0 \\ v(T, x) = u(x) \end{cases}$$

$$\implies \begin{cases} v'_t + \sup_{\alpha\in\mathcal{A}} \left\{ (\alpha\mu + (1 - \alpha)r)xv'_x + \frac{1}{2}\alpha^2\sigma^2x^2V''_{xx} \right\} = 0 \\ v(T, x) = u(x) \end{cases}$$

First let's calculate the supremum over $\alpha$ and denote $y := v'_x, z = v''_{xx}$ for simplicity:

$$H(x, y, z) = \sup_{\alpha \in \mathbb{R}} \left\{ (\alpha\mu + (1-\alpha)r)xv'_x + \frac{1}{2}\alpha^2\sigma^2x^2v''_{xx} \right\}$$

This is a parabola over $\alpha$, if $z < 0$, maximum is attained at the top of parabola at $\alpha = \frac{-(\mu-r)y}{\sigma^2 xz}$, and if $z \geq 0$, maximum is $+\infty$. In other words,

$$H(x, y, z) = \begin{cases} rxy - \frac{(\mu-r)^2y^2}{2\sigma^2 z}, & \text{if } z < 0 \\ +\infty, & \text{if } z \geq 0 \end{cases}$$

Let's assume for now that our solution satisfies $z < 0$ (we will check this later) and plug found $H$ into HJB equation:

$$\begin{cases} v'_t + rxy - \frac{(\mu-r)^2(v'_x)^2}{2\sigma^2 v''_{xx}} \\ v(T, x) = u(x) := \frac{x^p}{p} \end{cases}$$

Let's look for the solution in the form $v(t, x) = f(x)\phi(t)$.
From boundary condition we have $v(T, x) = f(x)\phi(T) = u(x) = \frac{x^p}{p}$, hence $f(x) = \frac{x^p}{p}$ and $\phi(T) = 1$. To find $\phi(t)$ we plug the form of $v(t, x)$ into the equation:

$$\phi'(t)\frac{x^p}{p} + rx\phi(t)x^{p-1} - \frac{(\mu-r)^2x^{2p-2}}{2\sigma^2(p-1)x^{p-2}}\phi(t) = 0$$

Hence

$$\begin{cases} \phi'(t) = \phi(t)p\left(\frac{(\mu-r)^2}{2\sigma^2(p-1)} - r\right) \\ \phi(T) = 1 \end{cases}$$

If we denote by $\boxed{\gamma := \frac{(\mu-r)^2}{2\sigma^2(p-1)} - r}$, then we will have $\phi(t) = e^{\gamma(t-T)}$ and hence

$$\boxed{v(t, x) = e^{\gamma(t-T)}\frac{x^p}{p}}$$

Finally, we need to check that $v''_{xx} < 0$: this is true due to the fact that $p \in (0, 1)$:

$$v''_{xx} = e^{\gamma(t-T)}(p-1)x^{p-2} < 0$$

And we also note that the optimal control is

$$\boxed{\hat{\alpha}_t = \frac{-(\mu-r)y}{\sigma^2 xz} = \frac{\mu-r}{\sigma^2(1-p)} > 0}$$

So surprisingly the optimal way of trading is to have a non-changing-in-time fraction of capital invested in the risky asset regardless of the time and amount of capital!

Here we show the graph of utility of terminal wealth for $\mu = 0.3, r = 0.2, \sigma = 0.5, T = 10, p = 0.5, S_0 = 50, B_0 = 1, X_0 = 100$.

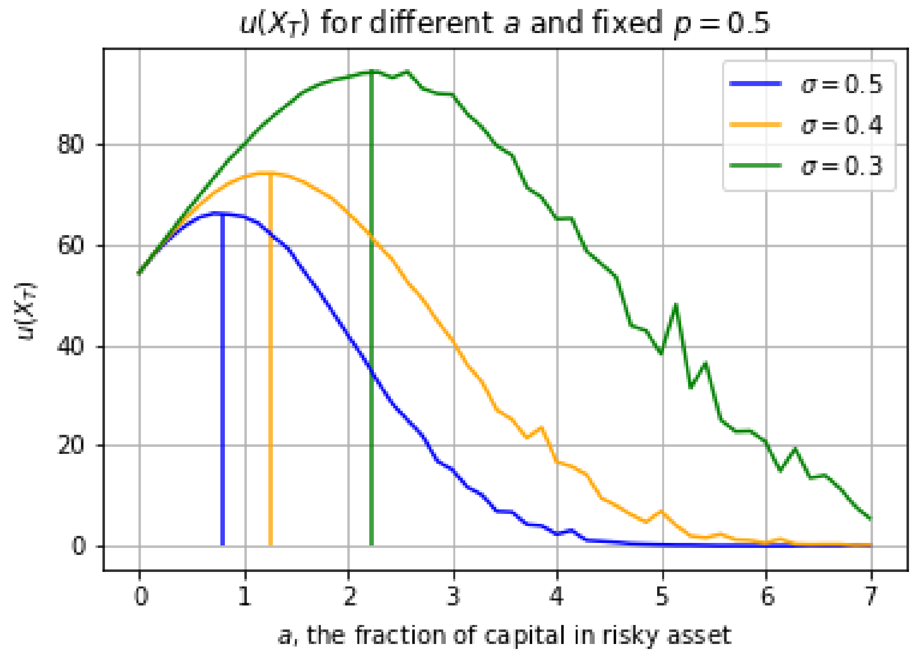


Figure 1: Utility of terminal wealth for $\mu = 0.3, r = 0.2, \sigma = 0.5, T = 10, p = 0.5, S_0 = 50, B_0 = 1, X_0 = 100$

### 3.1.2 Solution by Pontryagin's stochastic maximum method

The problem can be written as:

$$\begin{cases} J(\alpha_.) = Eu(X_T) \to \max_\alpha \\ dX_t = (\alpha_t\mu + (1-\alpha_t)r)X_t dt + \alpha_t \sigma X_t dW_t \\ X_0 = y = const \\ u(X_T) = \frac{X_T^p}{p} \qquad (0 < p < 1 \text{ in order for utility function to be concave}) \end{cases}$$

First we write down the Hamiltonian:

$$H(t, x, a, y, z) = b \cdot y + Tr(\sigma^T z) + f = \boxed{(a\mu + (1-a)r)xy + \sigma axz}.$$

For optimality over $x$ we write the adjoint equation:

$$dy_t = -\frac{\partial H}{\partial x} dt + z_t dWt.$$

To write the adjoint equation in case of Merton problem, we need to compute $\frac{\partial H}{\partial x}$ and $z_t$. For $\frac{\partial H}{\partial x}$ just differentiate $H$:

$$\frac{\partial H}{\partial x} = (a\mu + (1-a)r)y + \sigma az$$

For $z_t$ we write Pontryagin's stochastic maximum principle and ask for the derivative of $H$ over $\alpha$ to be equal to zero:

$$\frac{\partial H}{\partial a} = (\mu - r)xy + \sigma xz = 0$$

$$=> \boxed{z = \frac{-(\mu - r)y}{\sigma}}$$

Now plug found $z_t$ and $\frac{\partial H}{\partial x}$ into the adjoint equation:

$$dY_t = -\frac{\partial H}{\partial x}dt + z_t dW_t = -\left((a\mu + (1-a)r)Y_t - \sigma a z_t\right)dt + z_t dW_t =$$

$$= -\left((a\mu + (1-a)r)Y_t - \sigma a\frac{-(\mu - r)Y_t}{\sigma}\right)dt - \frac{(\mu - r)Y_t}{\sigma}dW_t =$$

$$= -rY_t dt - \frac{(\mu - r)Y_t}{\sigma}dW_t.$$

So $Y_t$ is Geometrical Brownian Motion (GBM) with exact solution

$$Y_t = Y_0 \exp\left\{\left(-r - \frac{(\mu - r)^2}{2\sigma^2}\right)t - \frac{(\mu - r)}{\sigma}W_t\right\}$$

Note that from boundary condition we require that

$$Y_T = u'(X_T) = X_T^{p-1}$$

But $X_t$ is also GBM with exact solution (we will look for $\alpha_t \equiv a$):

$$X_t = X_0 \exp\left\{\left(a\mu + (1-a)r - \frac{a^2\sigma^2}{2}\right)t + a\sigma W_t\right\}$$

So from boundary condition $Y_T = X_T^{p-1}$ we obtain:

$$Y_0 e^{\left\{\left(-r - \frac{(\mu - r)^2}{2\sigma^2}\right)T - \frac{(\mu - r)}{\sigma}W_T\right\}} = X_0 e^{\left\{(p-1)\left(a\mu + (1-a)r - \frac{a^2\sigma^2}{2}\right)T + (p-1)a\sigma W_T\right\}}$$

$$=> -\frac{(\mu - r)}{\sigma} = \hat{a}\sigma(p-1)$$

$$=> \boxed{\hat{a} = \frac{(\mu - r)}{\sigma^2(1-p)}}$$

Note that the answer coincides with the one that we obtained by HJB equation approach.

## 3.2 Merton problem with consumption

We can slightly modify Merton problem: we allow the investor to consume the capital with rate $c_t$, also chosen by investor. This means that at each point of the horizon $[0, T]$ investor consumes $c_t X_t$ amount of money, which is counted (as in the previous example without consumption) with utility functional $u(x) = \frac{x^p}{p}, p \in (0, 1)$ for utility function to be concave. We allow one more parameter, $\beta$, to regulate, how much or less valuable for the investor is to consume at the terminal point of time $T$ rather than at times $t < T$.

### 3.2.1 Solution by Pontryagin's stochastic maximum principle

$$\begin{cases} J(a_.) = \mathbb{E}\left(\int_0^T \frac{(c_t X_t)^p}{p} dt + \beta \frac{X_T^p}{p}\right) \\ dX_t = (a_t\mu + (1-a_t)r - c_t)X_t dt + \sigma a_t X_t dW_t \end{cases}$$

Hamiltonian:

$$H(t,x,a,y,z,c) = (a\mu + (1-a)r - c)xy + \sigma axy + \frac{(cx)^p}{p}$$

We want to maximize $H$ over $a, c$:

$$\begin{cases} 1) H'_a = 0 : (\mu - r)xy + \sigma xz = 0 => \boxed{\hat{z} = -\frac{(\mu - r)y}{\sigma}} \\ 2) H'_c = 0 : -xy + c^{p-1}x^p = 0 => \boxed{\hat{c}^{p-1} = \frac{y}{x^{p-1}}} \end{cases}$$

Now write the equation for $Y_t$:

$$dY_t = -\hat{H}'_x dt + z_t dW_t$$

We can write down the derivatives of $H$:

$$\hat{H}'_x = (\hat{a}\mu + (1-\hat{a})t - \hat{c})y + \sigma\hat{a}z + \hat{c}^p x^{p-1} =$$

$$= (\hat{a}\mu + (1-\hat{a})t - \hat{c})y - (mu - r)\hat{a}y + \hat{c}^p x^{p-1} = (r - \hat{c})y + \hat{c}y = ry$$

Hence

$$dY_t = -rY_t dt - \frac{(\mu - r)Y_t}{\sigma_t} dW_t$$

It is a GBM with the following solution:

$$Y_t = Y_0 \exp\left\{-\frac{(\mu - r)}{\sigma} W_t - \left(r + \frac{(\mu - r)^2}{2\sigma^2}\right) t\right\}$$

Now we will look for constant $a_t \equiv a, c_t \equiv c$, and in this case $X_t$ is also a GBM with exact solution:

$$X_t = X_0 e^{\left\{\left(a\mu + (1-a)r - c\frac{a^2\sigma^2}{2}\right)T + a\sigma W_T\right\}}$$

And we want that $Y_T = u'(X_T) = \beta X_T^{p-1}$.
Hence we equate two solutions of GBM:

$$Y_0 e^{\left\{-\frac{(\mu - r)}{\sigma} W_T - \left(r + \frac{(\mu - r)^2}{2\sigma^2}\right)T\right\}} = \beta X_0^{p-1} e^{\left\{(p-1)\left(a\mu + (1-a)r - c - \frac{a^2\sigma^2}{2}\right)T + (p-1)a\sigma W_T\right\}}$$

Hence

$$-\frac{(\mu - r)}{\sigma} = (p-1)\sigma a$$

$$=> \boxed{\hat{a} = \frac{\mu - r}{(1-p)\sigma^2}}$$

Interestingly enough, the optimal strategy is the same as in the problem without consumption! And we can also find the optimal consumption:

$$\hat{c} = \frac{y^{\frac{1}{p-1}}}{x} = \beta.$$

# 4 Application 2: problem of beating a target

In this chapter we analyze the optimal portfolio and investment policy for an investor who is concerned about the performance of his wealth relative only to the performance of a particular benchmark. Specifically, we consider the case where a chosen benchmark evolves stochastically over time, and the investor's objective is to exceed the performance of this benchmark (in a sense to be made more precise later) by investing in other stochastic processes. We take as our setting the continuous-time framework pioneered by Merton in Merton (1971). The portfolio problem where the objective is to exceed the performance of a selected target benchmark is sometimes referred to as active portfolio management, see for example Sharpe et al. (1995). It is well known that many professional investors in fact follow this benchmarking procedure: for example, many mutual funds take the Standard and Poors 500 Index as a benchmark, commodity funds seek to beat the Goldman Sachs Commodity Index, so the problem of beating a moving target is very actual nowadays. This problem was studied by many authors and different aspects were considered, see, for example, Browne (1991). In this work we will formulate the general framework but will focus on building the machinery for explicitly solving the problem for constant benchmark. Our work follows Chapter 4 of Wang (2024), but our extension of that model is the introduction of non-zero risk-free rate into the market.

## 4.1 General problem setup with zero risk-free rate

We are given a probability space $(\Omega, F, P)$, supporting an $(N+1)$-dimensional Brownian motion $W_t = (W_t^1, ..., W_t^{N+1})$. The probability space is endowed with the natural filtration $F = (F_t), t \geq 0$ generated by the Brownian motion $W$, with $F_t := \sigma\{W_s : 0 \leq s \leq t\}$. Assume that there are $N$ risky assets whose prices at time t are denoted by $\{S_t^{(i)}\}_{i=1}^N$, and one riskless bond offering zero interest rate in the market, which serves as the numeraire. The risky stock prices satisfy

$$dS_t^i = \mu_t^i S_t^i dt + \sum_{j=1}^N \sigma_t^{ij} S_t^i dW_t^j, \qquad i = 1, \dots, N.$$

By introducing column vectors $\mu_t = (\mu_t^1, ..., \mu_t^n)^T, S_t = (S_t^1, ..., S_t^N)^T$, and matrix $\sigma_t = (\sigma_t)^{ij}$, we can write:

$$dS_t = diag(S_t)(\mu_t dt + \sigma_t dW_t)$$

We also introduce the unique market price of risk $\kappa_t = \sigma^{-1}\mu_t$ by solving equation $\sigma_t \kappa_t = \mu_t$.

Trading strategies are described by means of progressively measurable vector processes $f = (f_t)_{t\geq 0} \in \mathbb{R}^N$ , where $f_t = (f_t^1, \dots, f_t^N)^T$ and $f_t^i$ denotes the fraction of wealth invested in the i-th risky asset at time t for $i = 1, \dots, N$, the remainder $\sum_{i=1}^N f_t^i$ is invested in the riskless asset. Let $X^f = (X_t^f)_{t\geq 0}$ denote the wealth of the investor following policy $f$ with initial wealth $X_0 = x > 0$, which is governed by the controlled stochastic differential equation

$$dX_t^f = X_t^f \left( \sum_{i=1}^N f_t^i \frac{dS_t^i}{S_t^i} \right) = X_t^f \left( \sum_{i=1}^N f_t^i \mu_t^i dt + \sum_{i=1}^N \sum_{j=1}^N f_t^i \sigma_t^{ij} dW_t^j \right) =$$

$$= X_t^f (f_t^T \mu_t dt + f_t^T \sigma_t d\overline{W_t}) = X_t^f f_t^T \sigma_t (\kappa_t dt + d\overline{W_t})$$

where $d\overline{W_t} = \left(W_t^1, \ldots, W_t^N\right)^T$.

The price of the goal $(Y_t)_{t\geq 0}$ is assumed to evolve according to a log-normal process, which is only partially correlated with the wealth process $X_f$, and is given by:

$$dY_t = \alpha Y_t dt + \sum_{i=1}^{N} b_i Y_i dW_t^i + \beta Y_t dW_t^{N+1} = \alpha Y_t dt + Y_t b^T d\overline{W_t} + \beta Y_t dW_t^{N+1}$$

We herein work with the ratio process $Z_t^f := X_t^f / Y_t$, which is also a controlled diffusion process and evolves according to SDE (obtained by applying Ito formula to $f(x,y) = \frac{x}{y}$):

$$dZ_t^f = Z_t^f(\hat{\eta} + f_t^T \hat{\mu}_t)dt + Z_t^f(f_t^T \sigma_t - b^T)d\overline{W_t} - Z_t^f \beta dW_t^{N+1}$$

with $\hat{\eta} = -\alpha + b^t b + \beta^2$ and $\hat{\mu}_t = \mu_t - \sigma_t b$.

We denote the discount function by $\rho : \mathbb{R}^+ \to (0,1]$. By assumption, it is a continuously differentiable and strictly decreasing function satisfying $\rho(0) = 1$ and $\lim_{t\to\infty} \rho(t) = 0$. This function, which maps the date $t$ to the discount factor $\rho(t)$, captures the agent's preferences concerning the timing of goal achievement, called time risk preferences.

Consider in our problem a fixed terminal time $T \in (0, +\infty]$ that is allowed to take the extended real numbers, and a real continuous function $u(z) : (0,1] \to (0,1]$ that is increasing, concave, and satisfies $\lim_{z\to 0} u(z) = 0$ and $u(1) = 1$. Given any initial data $(s, Z_s = z) \in [0,T) \times (0,1)$, we control up to the smaller of $T$ and the exit time of process $Z_t^f, t > s$ from the fixed domain $O = (0,1)$. Let $\hat{T}_s^f = \min(\tau_s^f, T)$, where $\tau_s^f := \inf\{t > s : Z_t^f \notin O\}$.

The reward functional is defined by

$$\hat{J}(s,z,f) := \mathbb{E}_{s,z}\left[1_{\{\tau_s^f \leq T\}}\rho(\tau_s^f) + 1_{\{\tau_s^f > T\}}\rho(T)u(Z_T^f)\right] = \mathbb{E}_{s,z}\left[\rho(\hat{T}_s^f) + \rho(T)u(Z_{\hat{T}_s^f}^f)\right] - \rho(T),$$

measuring the expected discounted rewards at the target-debut time before terminal horizon $T$, starting from state $Z_s = z$ at time $s$ if control process $f$ is implemented.

We observe time inconsistency in optimization problems with this objective functional, due to the dependence of $\tau_s^f$ on the initial time point $s$. To address this issue, we reformulate the reward functional by first introducing a 2-dimensional degenerate diffusion process $N_r^f = (M_r, Z_r^f)$ for $r \geq 0, N_0 = (s,z)$, that evolves according to

$$\begin{cases} dZ_t^f = Z_t^f(\hat{\eta} + (f_r \circ \theta_s)^T(\hat{\mu}_r \circ \theta_s))dr + Z_r^f((f_r \circ \theta_s)^T(\sigma_r \circ \theta_s)^T(\sigma_r \circ \theta_s) - b^T)d\overline{W_r} - Z_r^f \beta d_r^{N+1} \\ dM_r = dr \end{cases}$$

where shift operator $\theta$ is defined as follows: for any fixed $s \in \mathbb{R}^+$, let $Z_t \circ \theta_s := Z_{t+s}$ for $t \in \mathbb{R}^+$, and $\tau_0^f \circ \theta_s = \inf t \geq 0 : Z_t \circ \theta_s \notin O$. Then we have $\tau_s^f = \tau_0^f \circ \theta_s + s$. Let $G = [0,T) \times (0,1)$ and $\tau_G^f := \inf\{t \geq 0 : N_t^f \notin G\} \leq T$.

The new reward functional is the following:

$$J(N_0, f) = \mathbb{E}_{N_0}\left[\int_0^{\tau_G^f} \rho(M_r)' dr + \rho(T)u(Z_{\tau_G^f}) - \rho(T)\right]$$

The reward functional $\hat{J}(N_0, f)$ in terms of $(N_r)_{r\geq 0}$ with $N_0 = (s,z)$ can be expressed by

$$\hat{J}(N_0, f) = \mathbb{E}_{s,z}\left[\int_s^{\hat{T}_s^f} \rho(r)' dr + \rho(T)u(Z_{\hat{T}_s^f}^f)\right] + \rho(s) - \rho(T) = J(N_0, f) + \rho(s)$$

With this transformation, our focus now becomes studying a standard time-homogeneous stochastic control problem with degenerate controlled states. Specifically, we aim to find for any $N_0 \in G$, the value of $V(N_0) = \sup_f J(N_0, f)$ and the corresponding optimal control $f^*$. Subsequently, we can apply the standard machinery to derive the HJB equation satisfied by $V(N_0)$ in this time-homogeneous problem, and ensure that the corresponding verification theorem holds. The value function of the original problem can be then determined by the relationship $\hat{V}(N_0) = V(N_0) + \rho(s)$.

Let $C^{1,2}(G)$ denote the set of real-valued functions that are continuous on the fixed domain $G$ and have continuous first-order partial derivatives with respect to s, as well as continuous first- and second-order partial derivatives with respect to $z$. Given admissible control processes $f$, and functions $v(N_0) \in C^{1,2}(G)$ with $N_0 = (s, z)$, the infinitesimal generator is defined by

$$\mathcal{L}^f v(N_0) = v'_s(N_0) + \frac{1}{2}\left[(f^T\sigma - b^T)(f^T\sigma - b^T)^T + \beta^2\right] z^2 v''_{zz}(N_0) + (f^T\hat{\mu} + \hat{\eta}) z v'_z(N_0).$$

By the dynamics of the ratio process $Z_t^f$ and (Theorem 11.2.1 in Oksendal (2013)), one can write down the corresponding HJB equation, i.e., the value function $V(N_0) = V(s, z) = v^a(s, z)$, where $v^a(s, z)$ satisfies

$$\sup_f \left\{\mathcal{L}^f v^a(s, z)\right\} + \rho'(s) = 0,$$

or equivalently,

$$v_s^a + \sup_f \left\{\frac{1}{2}\left[(f^T\sigma - b^T)(f^T\sigma - b^T)^T + \beta^2\right] z^2 v_{zz}^a + (f^T\hat{\mu} + \hat{\eta}) z v_z^a\right\} + \rho'(s) = 0.$$

This equation is to be considered in the set $G$ with boundary data $v^a(s, 1) = 0$, since $v^a(s, z) = v(s, z) + \rho(s)$. Additionally, in the case of a finite terminal horizon $T$, one needs to further impose boundary condition on $s = T$, namely, $v^a(T, z) = \rho(T)(u(z) - 1)$.

We may then use standard calculus for optimization with respect to $f$ to obtain optimal policy $f^*$ in terms of the value function, in a feedback form,

$$f^*(s, z) = ((\sigma_s)^{-1})^T b - \Sigma_s^{-1}\hat{\mu}_s \frac{v_z^a(s, z)}{z v_{zz}^a(s, z)}, \qquad \text{where } \Sigma_s = \sigma_s \sigma_s^T.$$

Next, we substitute the feedback form of $f^*$ back into HJB equation, and obtain

$$\begin{cases} v_s^a(s, z) - \lambda(s)\frac{(v_z^a(s,z))^2}{v_{zz}^a(s,z)} + \delta(s) z v_z(s, z) + \frac{1}{2}\beta^2 z^2 v_{zz}^a(s, z) + \rho'(s) = 0; (s, z) \in [0, T) \times (0, 1) \\ v^a(s, 1) = 0 \\ v^a(T, z) = \rho(T)(u(z) - 1) \qquad \text{if } T < +\infty \end{cases}$$

where $\lambda(s) = 0.5\hat{\mu}_s^T \Sigma_s^{-1}\hat{\mu}_S$ and $\delta(s) = -\alpha + \beta^2 + b^T\sigma_s^{-1}\mu_s$.

Furthermore, if we denote the optimal value function $\hat{V}(s, z)$ of the original problem by $v(s, z)$, then according to the relationship between $\hat{V}$ and $V$, last equation can be equivalently transformed to

$$\begin{cases} v_s(s, z) - \lambda(s)\frac{(v_z(s,z))^2}{v_{zz}(s,z)} + \delta(s) z v_z(s, z) + \frac{1}{2}\beta^2 z^2 v_{zz}(s, z) = 0; (s, z) \in [0, T) \times (0, 1) \\ v(s, 1) = \rho(s) \\ v(T, z) = \rho(T) u(z) \qquad \text{if } T < +\infty \end{cases}$$

### 4.2 Infinite time horizon and constant benchmark

Now we study the case where
1) horizon is infinite and
2) the benchmark is a constant, that is $Y_t \equiv Y_0 > X_0$.

Then the ration process $Z_t^f := X_t^f / Y_0$ reduces to

$$dZ_t^f = Z_t^f f_t^T \mu_t dt + Z_t^f f_t^T \sigma_t d\overline{W_t} = Z_t^f f_t^T \sigma_t (\kappa_t dt + d\overline{W_t})$$

and the corresponding HJB equation reduces to

$$\begin{cases} v_s(s,z) - \lambda(s) \frac{v_z^2(s,z)}{v_{zz}(s,z)} = 0, z \in (0,1) \\ v(s,1) = \rho(s) \end{cases} \tag{14}$$

For computational purposes, we first solve this equation for $0 < z < 1$ with boundary data $v(s,1) = \hat{\rho}(s)$, where $\hat{\rho} : \mathbb{R}^+ \to \mathbb{R}^-$ is a continuously differentiable and strictly decreasing function with $\hat{\rho}(0) = 0$. Later, we will restrict our attention to bounded functions and proceed with normalization, ensuring that the boundary data $\rho(s)$ imposed at $z = 1$ for the original problem falls within the range $(0, 1]$, with $\rho(0) = 1$. As outlined in Misuela and Zariphopoulou (2010), there exists a one-to-one correspondence (modulo normalization constants) between strictly increasing functions $h(s,z)$ satisfying

$$h_s(s,z) + \frac{1}{2} h_{zz}(s,z) = 0 \tag{15}$$

and strictly increasing solutions to (14) with $\lambda(s) = \frac{1}{2}$:

$$v_s(s,z) - \frac{1}{2} \frac{v_z^2(s,z)}{v_{zz}(s,z)} = 0$$

**Lemma 1.** *(Form of solutions to equation (15))*
*If measure $\nu \in \mathcal{B}^+$, then*

$$h(s,z) = \int_{\mathbb{R}} \frac{e^{yz - \frac{1}{2} y^2 s} - 1}{y} \nu(dy) + C \tag{16}$$

*is a strictly increasing solution of (15):*

$$h_s(s,z) + \frac{1}{2} h_{zz}(s,z) = 0$$

**Lemma 2.** *(Expression of $v$ in terms of $h$)*
*Let $\Theta(s)$ be the antiderivative of $\lambda(s)$. The function $v(s,z), (s,z) \in \mathbb{R}^+ \times (0,1]$ is an increasing and strictly concave solution to equation (14) with boundary data $v(s,1) = \hat{\rho}(s)$ if and only if $v$ and the first derivative of $\hat{\rho}$ admit representation respectively given by*

$$\begin{cases} v(s,z) = -\frac{1}{2} \int_0^{2\Theta(s)} e^{-h^{(-1)}(t,z) + \frac{t}{2}} h_z\left(t, h^{(-1)}(t,z)\right) dt + \int_1^z e^{-h^{(-1)}(0,x)} dx \\ \hat{\rho}'(s) = -\lambda(s) e^{-h^{(-1)}(s\Theta(s),1) + \Theta(s)} h_z\left(t, h^{(-1)}(2\Theta(s),1)\right) \end{cases} \tag{17}$$

The associated boundary data $\hat{\rho}'$, along with its first-order derivative given by equation (17), is expressed as:

$$\hat{\rho}(s) = v(s,1) = -\lambda(s) \int_0^{2\Theta(s)} e^{-h^{(-1)}(t,1) + \frac{t}{2}} h_z\left(t, h^{(-1)}(t,1)\right) dt \tag{18}$$

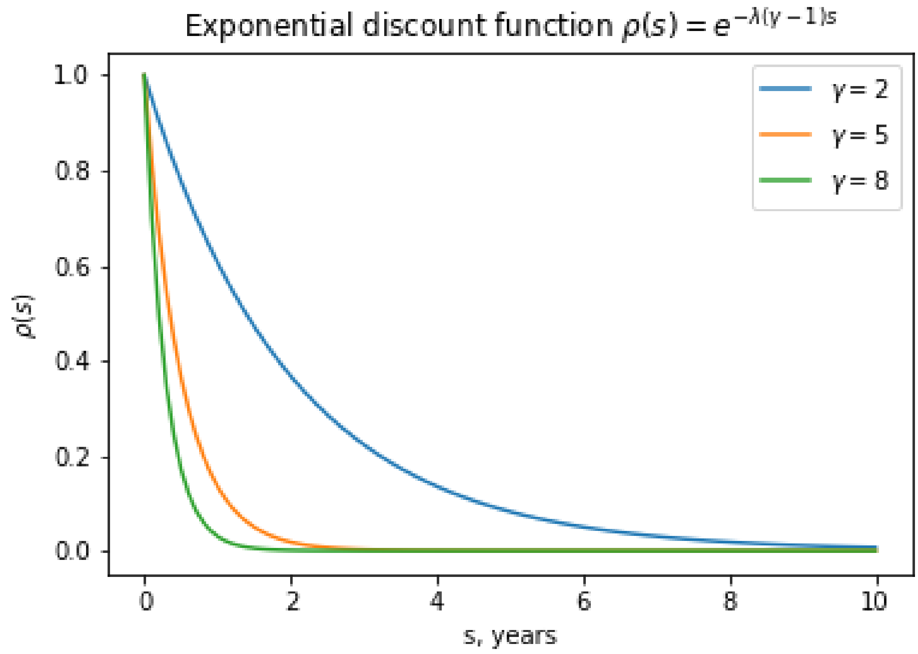


Figure 2: Exponential discount function $\rho(s) = e^{-\lambda(\gamma-1)s}$

with $\hat{\rho}(0) = 0$;

The optimal strategy given a constant benchmark can be further written, in terms of function $h$, by

$$f^*(s,z) = -\Sigma^{-1}\hat{\mu}_s\left(\frac{v_z(s,z)}{zv_z(s,z)}\right) = \Sigma^{-1}\hat{\mu}_s\frac{h_z\left(2\Theta(s), h^{(-1)}(2\Theta(s),z)\right)}{z}$$

**Lemma 3.** *(How for $\hat{\rho}$ to fulfill equation (17))*

*A function $\hat{\rho}: \mathbb{R}^+ \to \mathbb{R}^-$ admits the representation (17) for some measure $\nu$ with $h$ given by (16) if and only if there exists function $\hat{w}: \mathbb{R}^- \times R \to \mathbb{R}$ satisfying*

$$\begin{cases} \hat{w}_t(t,\hat{z}) = \hat{z}^2\hat{w}_{zz}(t,\hat{z}) \\ \hat{w}_z(-\Theta(s), \hat{w}^{(-1)}(-\Theta(s),1)) = -\frac{\hat{\rho}'(s)}{\lambda(s)} \\ \hat{w}_z(t, \hat{w}^{(-1)}(t,z)) > 0 \end{cases} \tag{19}$$

Finally, we need to make the normalization of $\hat{\rho}$ into $\rho$: $\rho(s) = -\frac{\hat{\rho}(s)}{\hat{\rho}(+\infty)} + 1$.

### 4.3 Example 1: exponential discounting

Consider the discount function $\rho(s) = e^{-\lambda(\gamma-1)s}$, $\gamma > 1$.

First let's summarize the steps to obtain the optimal strategy, and then (in Appendix) we will do these steps explicitly to demonstrate the machinery we have built.

First, such function $\rho(s)$ corresponds to $\hat{\rho}(s)$ defined by

$$\hat{\rho}(s) = \frac{\gamma^{\frac{\gamma-1}{\gamma}}}{\gamma-1}e^{-\lambda(\gamma-1)s} - \frac{\gamma^{\frac{\gamma-1}{\gamma}}}{\gamma-1}$$

Second, this $\hat{\rho}$ can be expressed as a solution of (18) with $h(s,z) = \frac{1}{\gamma}e^{\gamma z - \frac{1}{2}\gamma^2 s}$, that can be obtained from (16) by taking Dirac measure $\nu = \delta_\gamma$.

We can also check that, as said in Lemma 3, our $\hat{\rho}$ admits the solution of equation (19) by the function $\hat{w}(t, \hat{z}) = \frac{e^{\gamma(\gamma-1)t}}{\gamma} z^{\gamma}$.

Therefore, having $h$, we obtain by (17) (after finding $\rho$ and normalizing it) that equation (14) with boundary condition $v(s, 1) = \rho(s)$ can be solved by

$$v(s, z) = z^{\frac{\gamma-1}{\gamma}} e^{-\lambda(\gamma-1)s}$$

The optimal strategy is given by

$$\boxed{f^*(s, z) = \Sigma_s^{-1} \mu_s \gamma.}$$

## 4.4 Discussion

In other words, we have provided the explicit formula for optimal strategy in case of constant target and exponential discount function (which is the most common discount function in finance and insurance). This result therefore may have a direct application to the real world as a recommendation to a pension fund that is aiming to beat the target inflation rate.
The full derivation of this result can be seen in Appendix.

## 4.5 One extension: evolution of capital with presence of risk-free rate

In this section we will extend the model from Chapter 4 from Wang (2024) by introducing the bank account $B_t$, that grows with risk-free rate $r$: $dB_t = rB_t dt$. That means that the investor allocates the money, that he didn't spent on risky assets, into bank account. In other words, he invests fraction $1 - \sum_{i=1}^{N} f_t^i$ of his capital into bank account $B_t$. Note that that we require $\mu^i > r$, otherwise there is no reason to invest in risky asset at all. Then the evolution of capital will be:

$$dX_t^f = X_t^f \left( \sum_{i=1}^{N} f_t^i \frac{dS_t^i}{S_t^i} \right) + X_t^f (1 - \sum_{i=1}^{N} f_t^i) \frac{dB_t^i}{B_t^i} =$$

$$= X_t^f \left( \sum_{i=1}^{N} f_t^i \mu_t^i dt + (1 - \sum_{i=1}^{N} f_t^i) r dt + \sum_{i=1}^{N} \sum_{j=1}^{N} f_t^i \sigma_t^{ij} dW_t^j \right) =$$

$$= X_t^f (f_t^T \mu_t dt + (1 - \sum_{i=1}^{N} f_t^i) r dt + f_t^T \sigma_t d\overline{W_t})$$

Now for $Z_t := \frac{X_t^f}{Y_t}$ we have: $f'_x = \frac{1}{y}, f'_y = -\frac{x}{y}, f''_{xx} = 0; f''_{xy} = -\frac{1}{y^2}, f''_{yy} = \frac{2x}{y^3}$

$$=> df \left( \frac{X_t^f}{Y_t} \right) = \frac{X_t^f}{Y_t} \left( (f_t^T \mu_t + (1 - \sum_{i=1}^{N} f_t^i) r) dt + f_t^T \sigma_t d\overline{W_t} \right) =$$

$$= -\frac{X_t^f}{Y_t^2} Y_t \left( \alpha dt + b^T d\overline{W_t} + \beta dW_t^{N+1} \right) + 0 - \frac{1}{Y_t^2} X_t^f Y_t f_t^T \sigma_t b dt + \frac{X_t^f}{Y_t^3} Y_t^2 \left( bb^T + \beta^2 \right) =$$

$$= Z_t \left( f_t^T \mu_t + (1 - \sum_{i=1}^{N} f_t^i) r dt + f_t^T \sigma_t d\overline{W_t} \right) - Z_t \left( \alpha dt + b^T d\overline{W_t} + \beta dW_t^{N+1} \right) -$$

$$-Z_t f_t^T \sigma_t b_t dt + Z_t (bb^T + \beta^2) dt =$$

$$= Z_t\left((r-\alpha+bb^T+\beta^2)+f_t^T(\mu_t-\sigma_t b-\bar{1}r)\right)dt + Z_t\left(f_t^T\sigma_t - b^T\right)d\overline{w_t} - Z_t\beta s W_t^{N+1} =$$

$$= Z_t^f(\hat{\eta}+f_t^T\hat{\mu}_t)dt + Z_t^f(f_t^T\sigma_t - b^T)d\overline{W_t} - Z_t^f\beta dW_t^{N+1}$$

with $\hat{\eta} = r-\alpha+b^Tb+\beta^2$ and $\hat{\mu}_t = \mu_t - \sigma_t b - \bar{1}r$.

According to our theorem, the optimal strategy in this case is:

$$\boxed{f^*(s,z) = ((\sigma_s)^{-1})^T b - \Sigma_s^{-1}\hat{\mu}_s\frac{v_z^a(s,z)}{zv_{zz}^a(s,z)}, \qquad \text{where } \Sigma_s = \sigma_s\sigma_s^T.}$$

## 4.6 Discussion

The question of interest in this problem setting is the following: are there any discount functions $\rho(s), \rho(0)=1, \lim_{t\to+\infty}\rho(t)=0$ that lead to $f^*\equiv 0$, meaning that the optimal portfolio will consist only of riskless asset, with no investment in risky asset. In case of utility function theory, the answer is no. In our case of discount function theory, the answer is again no, at least in the case of constant $\mu_s, \sigma_s$.

First, if target is constant, which means $b=0, \alpha=0, \beta=0$, then $f^*(s,z) = \Sigma_s^{-1}\hat{\mu}_s\frac{v_z^a(s,z)}{zv_{zz}^a(s,z)}$ can be identically zero if and only if $v_z(s,z)\equiv 0$ and $zv''_{zz}(s,z)\neq 0$, which can't hold simultaneously.

Second, if target is not constant, in one-dimensional case we have:

$$f^*(s,z) = \frac{b}{\sigma_s} - \frac{\hat{\mu}_s}{\sigma_s^2}\frac{v'_z}{zv''_{zz}} = 0$$

$$v'_z = b\frac{\sigma_s}{\hat{\mu}_s}zv''_{zz}$$

Denoting $w := v'_z$, we obtain and solve the equation for $w$:

$$w = \frac{\sigma_s}{\hat{\mu}_s}zw'_z$$

$$w = Ce^{\frac{b\sigma_s}{\mu_s}z}, \qquad \text{for some constant } C.$$

$$v = \tilde{C}e^{\frac{b\sigma_s}{\mu_s}z}, \qquad \text{for some constant } \tilde{C}.$$

Then $\rho(s) = v(s,1) = \tilde{C}e^{\frac{b\sigma_s}{\mu_s}}$. It is evident that in case of constant $\mu_s, \sigma_s$ no choice of constant $\tilde{C}$ can normalize this $\rho(s)$ to satisfy the required conditions for $\rho(s)$: $\rho(0) = 1, \lim_{t\to+\infty}\rho(s) = 0$. Hence we conclude that in case of constant $\mu_s, \sigma_s$ there are no discount functions that lead to optimal strategy that will not invest some fraction of capital into the risky asset. The case of non-constant $\mu_s, \sigma_s$ needs further investigation.

Finally, we want to emphasize the obtained result in connection to stock market participation puzzle. The stock market participation puzzle refers to the fact that many people do not participate in the stock market, despite the potential benefits of investing in stocks for long-term financial growth.

The canonical Portfolio Model of Markowitz (1952) implies that all households should be holding some part of their investments in risky securities unless they are infinitely risk averse and/or expected equity risk premium is not present in the market (i.e. excess return over the risk-free asset is zero or negative). However, empirical evidence suggests that in the long-term investments in stocks earn positive equity premium and investors do not have unreasonably high level of risk aversion. And yet individuals are underinvesting in stocks in general and investment funds in particular. For example, in the United States, only about half of individuals own stock either directly or indirectly. Furthermore, participation rates are even lower outside the United States. This phenomenon is named as “the stock market participation puzzle” (see Mankiw and Zeldes (1991)). As we see, our last result as well suggests that all people with all discount-functions should invest some part of their capital in stock, but this behaviour does not coincide with the reality. The reasons for this puzzle are complicated and deserve a separate discussion.

## 5 Conclusion

In this project we considered two connected themes in the area of portfolio management.

First of all, in Chapter 2 we studied two principal and most commonly used approaches for solving stochastic optimal control problems: HJB equations approach and Pontryagin's stochastic maximum principle approach.

Then, in Chapter 3 we applied Pontryagin's stochastic maximum principle to tackle the famous Merton problem introduced in Merton (1969), that, as far as we know, has never been solved this way. We obtained the same optimal control, as in HJB approach, and, interestingly enough, this optimal control is constant, that is, the investor should invest the same fraction of his capital in the risky asset at any time of the investment horizon.

Finally, in Chapter 3 we extended the theory of time risk-preferences with different discount functions from Chapter 4 of Wang (2024) by introducing risk-free rate into the model. We presented the machinery for tackling the problem in general case and in the case of exponential discounting (which is the most common discounting function both in financial and insurance industries) we gave the exact formula for optimal strategy. This result is of a great interest for both financial and insurance industries, since insurance companies and pension funds in their everyday work follow a benchmarking procedure, for example by trying to beat inflation, exchange rates, or other indices. We also showed that in case of presence of risk-free rate and constant $\mu_s, \sigma_s$ there are no discount functions that lead to the optimal strategy not investing in the risky asset at all. This result is a bit counter-intuitive, since in real world we observe that the majority of households prefer to invest only in the bank account. This paradox is referred to as “the stock market participation puzzle”.

For more applications and other variations of stochastic target problems one may refer to Chapter 8 of Touzi (2013).

## 6 Appendix

Here we carry all the steps of obtaining the value function in case of exponential discounting in order to demonstrate, how works our proposed technique. According to (15),

$$h(s,z) = \int_{\mathbb{R}} \frac{e^{yz-\frac{y^2 s}{2}}-1}{\gamma}\nu(dy) + C$$

We want to take Dirac measure, sitting at point gamma: $\nu := \delta_\gamma$.

Then

$$h(s,z) = \frac{e^{\gamma z-\frac{\gamma^2 s}{2}}}{\gamma}$$

Then, according to (17),

$$\hat{\rho}'(s) = -\lambda(s) e^{-h^{(-1)}(\Theta(s),1)+\Theta(s)} h_z\left(t, h^{(-1)}(2\Theta(s),1)\right)$$

So to obtain $\hat{\rho}'(s)$, we need to compute $h'_z$, $h^{(-1)}$ and $\Theta(s)$.
We have spatial derivative of $h$:

$$\boxed{h'_z(s,z) = e^{\gamma z - \frac{\gamma^2 s}{2}}}$$

And spatial inverse of h:

$$h(s,z) = \frac{e^{\gamma z - \frac{\gamma^2 s}{2}}}{\gamma}$$

$$=> \ln(\gamma h(s,z)) = \gamma z - \frac{\gamma^2 s}{2}$$

$$=> z = \frac{\ln(\gamma h(s,z)) + \frac{\gamma^2 s}{2}}{\gamma}$$

$$=> \boxed{h^{(-1)}(s,u) = \frac{\ln(\gamma u) + \frac{\gamma^2 s}{2}}{\gamma}}$$

We also need antiderivative of $\lambda$:

$$\lambda(s) = \frac{1}{2} => \boxed{\Theta(s) = \int_0^s \lambda(s) ds = \frac{s}{2}}$$

Now we can compute the three components in the formula of $\hat{\rho}(s)$:

$$h^{(-1)}(2\Theta(s),1)) = \frac{\ln(\gamma u) - \frac{\gamma^2 s}{2}}{\gamma}(s = s, u = 1) = \boxed{\frac{\ln\gamma - \frac{\gamma^2 s}{2}}{\gamma}}$$

And

$$h'_z\left(t, h^{(-1)}(2\Theta(s),1)\right) = e^{\gamma z - \frac{\gamma^2 s}{2}}\left(s = t, z = \frac{\ln\gamma - \frac{\gamma^2 s}{2}}{\gamma}\right) = e^{\ln\gamma} = \boxed{\gamma}$$

And

$$e^{-h^{(-1)}(\Theta(s),1)+\Theta(s)} = e^{-\frac{\ln\gamma - \frac{\gamma^2 s}{2} + \frac{s}{2}}{\gamma}} = e^{\frac{-\ln\gamma - \frac{\gamma(\gamma-1)s}{2}}{\gamma}} = e^{\frac{-\ln\gamma}{\gamma}} e^{-\frac{(\gamma-1)}{2}} = \left(\frac{1}{\gamma}\right)^{\frac{1}{\gamma}} e^{-\frac{(\gamma-1)}{2}} = \gamma^{-\frac{1}{\gamma}} e^{-\frac{(\gamma-1)s}{2}}$$

So finally we can compute the derivative of $\hat{\rho}(s)$:

$$\hat{\rho}'(s) = -\lambda(s) e^{-h^{(-1)}(\Theta(s),1)+\Theta(s)} h_z\left(t, h^{(-1)}(2\Theta(s),1)\right) = -\frac{1}{2}\gamma^{-\frac{1}{\gamma}} e^{-\frac{(\gamma-1)s}{2}} \cdot \gamma = \boxed{-\frac{1}{2}\gamma^{\frac{\gamma-1}{\gamma}} e^{-\frac{(\gamma-1)s}{2}}}$$

Integrating this function, we obtain $\hat{\rho}(s)$:

$$\hat{\rho}(s) = \frac{\gamma^{\frac{\gamma-1}{\gamma}}}{\gamma - 1} e^{-\lambda(\gamma-1)s} - \frac{\gamma^{\frac{\gamma-1}{\gamma}}}{\gamma - 1}$$

The derivative of this function is (taking into account that $\lambda(s) = \frac{1}{2}$:

$$\hat{\rho}'(s) = -\lambda(s)\gamma^{\frac{\gamma-1}{\gamma}} e^{-\lambda(\gamma-1)s} = \boxed{-\frac{1}{2}\gamma^{\frac{\gamma-1}{\gamma}} e^{-\frac{(\gamma-1)s}{2}}}$$

So we have verified that if we take as $\nu$ Dirac measure $\delta_\gamma$, then we have

$$h(s,z) = \frac{e^{\gamma z - \frac{\gamma^2 s}{2}}}{\gamma}$$

and

$$\hat{\rho}'(s) = -\frac{1}{2}\gamma^{\frac{\gamma-1}{\gamma}} e^{-\frac{(\gamma-1)s}{2}}$$

and hence

$$\hat{\rho}(s) = \frac{\gamma^{\frac{\gamma-1}{\gamma}}}{\gamma-1} e^{-\lambda(\gamma-1)s} - \frac{\gamma^{\frac{\gamma-1}{\gamma}}}{\gamma-1}$$

and hence

$$\rho(s) = -\frac{\hat{\rho}(s)}{\hat{\rho}(+\infty)} + 1 = e^{-\lambda(\gamma-1)s}$$

And now, after obtaining $\hat{\rho}'(s)$, in order to obtain $v$, we need to solve

$$\begin{cases} \hat{w}_t(t,\hat{z}) = \hat{z}^2 \hat{w}_{zz}(t,\hat{z}) \\ \hat{w}_z(-\Theta(s), \hat{w}^{(-1)}(-\Theta(s),1)) = -\frac{\hat{\rho}'(s)}{\lambda(s)} \\ \hat{w}_z(t, \hat{w}^{(-1)}(t,z)) > 0 \end{cases} \tag{20}$$

The solution is claimed to be

$$w(t,z) = \frac{e^{\gamma(\gamma-1)t}}{\gamma} z^\gamma$$

It clearly satisfies $w'_t = z^2 w''_{zz}$
And

$$\boxed{w'_z(t,z) = e^{\gamma(\gamma-1)t} z^{\gamma-1}}$$

Boundary condition is:

$$\boxed{-\frac{\hat{\rho}'(s)}{\lambda(s)} = \gamma^{\frac{\gamma-1}{\gamma}} e^{-\frac{(\gamma-1)s}{2}}}$$

Let's obtain the formula for $w^{-1}$:

$$w(t,z) = \frac{e^{\gamma(\gamma-1)t}}{\gamma} z^\gamma$$

$$=> z^\gamma = e^{-\gamma(\gamma-1)t} \gamma w(t,z)$$

$$=> z = e^{-(\gamma-1)t} \gamma^{\frac{1}{\gamma}} w(t,z)^{\frac{1}{\gamma}}$$

$$=> \boxed{w^{(-1)}(t,w) = e^{-(\gamma-1)t} \gamma^{\frac{1}{\gamma}} w^{\frac{1}{\gamma}}}$$

$$=> w^{(-1)}(-\Theta(s),1) = e^{\frac{(\gamma-1)s}{2}} \gamma^{\frac{1}{\gamma}}$$

$$=> w'_z(-\Theta(s), w^{(-1)}(-\Theta(s),1)) = w'_z(-\frac{1}{2}, e^{\frac{(\gamma-1)s}{2}} \gamma^{\frac{1}{\gamma}}) =$$

$$= e^{\gamma(\gamma-1)\cdot(-\frac{s}{2})} \cdot \left(e^{\frac{(\gamma-1)s}{2}}\gamma^{\frac{1}{\gamma}}\right)^{(\gamma-1)} =$$

$$= e^{-\frac{s\gamma(\gamma-1)}{2}} e^{\frac{s(\gamma-1)^2}{2}} \gamma^{\frac{\gamma-1}{\gamma}} = e^{-\frac{s(\gamma-1)(\gamma-(\gamma-1))}{2}} \gamma^{\frac{\gamma-1}{\gamma}} = \boxed{e^{-\frac{(\gamma-1)s}{2}} \gamma^{\frac{\gamma-1}{\gamma}}}$$

This coincides with needed boundary condition $-\frac{\hat{\rho}'(s)}{\lambda(s)}$.

Now we have checked that our $\hat{\rho}(s)$ admits representation via $w$ and hence this $\hat{\rho}(s)$ is good and we can use this $\hat{\rho}(s)$ (and its corresponding $h$) to calculate $v(s,z)$:

$$v(s,z) = -\frac{1}{2}\int_0^{2\Theta(s)} e^{-h^{(-1)}(t,z)+\frac{t}{2}} h_z\left(t, h^{(-1)}(t,z)\right) dt + \int_1^z e^{-h^{(-1)}(0,x)} dx$$

We have previously computed that for our $\hat{\rho}$:

$$\boxed{h^{(-1)}(s,u) = \frac{\ln(\gamma u) + \frac{\gamma^2 s}{2}}{\gamma}}$$

and

$$\boxed{h'_z(s,z) = e^{\gamma z - \frac{\gamma^2 s}{2}}}$$

So

$$h'_z(t, h^{(-1)}(t,z)) = e^{\gamma z - \frac{\gamma^2 s}{2}} \left(s = t, z = \frac{\ln(\gamma z) + \frac{\gamma^2 t}{2}}{\gamma}\right) = e^{\ln(\gamma z) + \frac{\gamma^2 t}{2} - \frac{\gamma^2 t}{2}} = \boxed{\gamma z}$$

Now compute $v(s,z)$:

$$1) -\frac{1}{2}\int_0^s e^{-h^{(-1)}(t,z)+\frac{t}{2}} h_z\left(t, h^{(-1)}(t,z)\right) dt = -\frac{1}{2}\int_0^s e^{\frac{\ln(\gamma z)}{\gamma} - \frac{\gamma t}{2} + \frac{t}{2}} (\gamma z) dt =$$

$$= -\frac{1}{2}(\gamma z)\left(\frac{1}{\gamma z}\right)^{\frac{1}{\gamma}} \int_0^s e^{\frac{t(1-\gamma)}{2}} dt = -\frac{1}{2}(\gamma z)\left(\frac{1}{\gamma z}\right)^{\frac{1}{\gamma}} \frac{2}{1-\gamma}(e^{\frac{s(1-\gamma)}{2}} - 1) =$$

$$= \boxed{\frac{\gamma^{\frac{\gamma-1}{\gamma}}}{\gamma - 1} z^{\frac{\gamma-1}{\gamma}} (e^{\frac{s(1-\gamma)}{2}} - 1)}$$

$$2) \int_1^z e^{-h^{(-1)}(0,x)} dx = \int_1^z e^{-\frac{\ln(\gamma x)}{\gamma}} dx = \int_1^z \left(\frac{1}{\gamma x}\right)^{\frac{1}{\gamma}} dx =$$

$$= \gamma^{-\frac{1}{\gamma}} \int_1^z x^{-\frac{1}{\gamma}} dx = \gamma^{-\frac{1}{\gamma}} \frac{z^{-\frac{1}{\gamma}+1} - 1}{-\frac{1}{\gamma}+1} = \boxed{\frac{\gamma^{\frac{\gamma-1}{\gamma}}}{\gamma-1}(z^{\frac{\gamma-1}{\gamma}} - 1)}$$

And finally

$$v(s,z) = \frac{\gamma^{\frac{\gamma-1}{\gamma}}}{\gamma-1} z^{\frac{\gamma-1}{\gamma}} (e^{\frac{s(1-\gamma)}{2}} - 1) + \frac{\gamma^{\frac{\gamma-1}{\gamma}}}{\gamma-1}(z^{\frac{\gamma-1}{\gamma}} - 1) = \boxed{\frac{\gamma^{\frac{\gamma-1}{\gamma}}}{\gamma-1}(z^{\frac{\gamma-1}{\gamma}} e^{\frac{s(1-\gamma)}{2}} - 1)}$$

Finally, going back to the initial problem by normalizing $\rho(s)$ to be bounded in $[0,1]$ with $\rho(0) = 0$ and $\lim_{t\to+\infty}\rho(t) = 0$ we obtain:

$$v(s,z) = z^{\frac{\gamma-1}{\gamma}} e^{-\lambda(\gamma-1)s}$$

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