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Stochastic control methods for optimization problems in Ornstein-Uhlenbeck spread models ☆

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ABSTRACT

We develop stochastic optimal control methods for spread financial models defined by the Ornstein–Uhlenbeck (OU) processes. To this end, we study the Hamilton - Jacobi - Bellman (HJB) equation using the Feynman - Kac (FK) probability representation. We show an existence and uniqueness theorem for the classical solution of the HJB equation, a quasi-linear partial derivative equation of parabolic type. Then we show a special verification theorem and, as a consequence, construct optimal consumption/investment strategies for power utility functions. Moreover, using fixed point tools we study the numeric approximation for the HJB solution and we establish the convergence rate which, as it turns out in this case, is super geometric, i.e., more rapid than any geometric one. Finally, we illustrate numerically the behavior of the obtained strategies.

1. Introduction

1.1. Motivations

This paper studies the optimal investment and consumption problem for spread models during a fixed time interval $[0, T]$. In this case, we do not define the individual dynamics of the asset itself but the difference (spread) between assets so that the underlying idea is to take long/short positions to get profit from deviations in the asset valuations. This approach has been proposed for nearly three decades in Wall

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Street by Nunzio Tartaglia's quantitative group at Morgan Stanley investment bank and financial services company. The spreads are used also to study precious metals markets, oil markets, electricity and gas markets, etc. (see, for example, [19,8,4] and the references therein). However, although the idea of pairs trading is widely used, academic researches are still small [7]. The spread (or pairs trading) strategy aims to gain profits from misprizing of two assets. This means to go long for one asset and go short on the other depending on the deviation from their long-term mean. By doing this consistently, the investor will gain profit from the opportunities generated by the divergence. To use the spread technique, assets must be jointly co-integrated, which means that the difference in their prices must be described by a stationary process, usually, an autoregressive process in discrete time, which corresponds to a stable Ornstein-Uhlenbeck (OU) process in continuous time (see, for example, in [1] for the details). This model is a particular case of the affine processes widely used in risky asset modeling [6,11]. It should be noted here that, there exist other spread models based on the Brownian bridge process (see, for example, [16]) but with additional conditions on the markets.

1.2. Main contribution

In this paper, we develop finance portfolio optimization methods for power utility functions for the OU spread markets. Firstly such problems were considered separately only for pure optimal investment or only for a special optimal consumption (see, for example in [3,9]). Unfortunately, in these papers, the authors find strategies that only maximize the main part in Hamilton–Jacobi–Bellman (HJB) equation. Usually, these strategies are optimal, but this should be proven in each case. In addition, which is most important for financial applications, it is necessary to find conditions under which these strategies will be optimal. The HJB maximization is not sufficient, this is just a necessary condition, but to construct optimal strategies one needs to maximize the objective functions not only the HJB equation. This means that the question of constructing optimal strategies for spread markets has not been studied even for pure optimal investment problems without consumption. In this paper, we study optimal investment and consumption problems in the classical mathematical economics setting and, probably for the first time, on the basis of the stochastic programming approach we give the solution for these problems in the complete form through the verification theorem method. The challenge here is that we could not use the HJB analysis method from [3,9] which was due to the additional strongly nonlinear term corresponding to the consumption. Moreover, it turns out that contrary to the Black–Scholes market, the HJB equation for the spread model has an additional variable corresponding to the risky asset. It should be noted, that the same situation is observed in the stochastic volatility case when the HJB equation contains additional variables corresponding to the volatility random generator. It should be noted that in stochastic volatility markets the HJB solution is bounded with respect to the additional variables and can be studied directly using the variable separation technique (see for example, [21,2,13]). For spread models the HJB solution is unbounded, moreover, it has an explosive exponential form and, unfortunately, cannot be studied by the methods previously developed for financial markets. In order to find optimal financial strategies in this paper, we develop special analytical methods to find conditions for market models which provide the uniform integrability property for the HJB solution calculated on the optimal wealth process (see, Condition **D**) in Section 3). It should be noted that this is the main condition to construct optimal strategies using verification theorem methods. In this paper, we study the HJB equation through the corresponding Feynman – Kac (FK) mapping and we show that the HJB solution coincides with the fixed point of this mapping in a special metric space in which this mapping is contracted. Using this property, we study the convergence rate of numerical methods for calculating optimal strategies.

1.3. Main tools

First note, that even in the pure investment problem, (see, for example, in [3]) the HJB solution is extremely explosive, i.e. it goes to infinity in a squared exponential power rate ($\approx e^{\mathbf{b}s^2}$) as the spread variable ($s \to \infty$) for some $\mathbf{b} > 0$. This means that if we replace the variable s with a gaussian random variable, we can obtain a nonintegrable random variable which can block the construction of the optimal strategies. It should be noted here, that there exist some special cases of the financial markets for the pairs trading where the HJB solution has not explosive form (see, for example, in [15,17,20]) and for this case, one can use the verification theorem methods developed for the Black − Scholes models. In this paper, we study the explosive HJB solution and we find conditions under which we provide the portfolio optimization solution for the power utility functions. In this case, the optimal strategies depend on the solution of a nonlinear partial derivative equation of parabolic type. We study this equation through the Feynman–Kac (FK) representation. It should be noted that for the first time this method was proposed in [5] for the optimal investment and consumption problem for the stochastic volatility markets with jumps. It turns out that the HJB solution coincides with the fixed point of the corresponding FK mapping. Later in [2] and [13] this approach was used for the Black-Scholes models with random coefficients and, moreover, it was found the convergence rate of the approximations for the fixed point which is super - geometric. This property is very important for practical implementations. Unfortunately, we can't use these results here since we can't represent directly HJB solution through the FK mapping. For spread models, the HJB solution has an exponential form of a solution of some quasi-linear equation of parabolic type which can be found through a corresponding probability representation, i.e. FK mapping, which is completely different from the FK mappings used early for optimization problems. For spread models, this mapping is much more complicated and, in particular, it depends nonlinearly on the partial derivatives of functions. Therefore, one needs to develop new special analytic tools to study FK mapping. To this end, we introduce a special metric space in which it is contracted. Taking this into account, we show the fixed point theorem for this mapping and we show that the FK fixed point coincides with the classical solution for the HJB equation in our case. Then we find an explicit upper bound for the approximation accuracy of the constructed iterative sequences and we get the convergence rate for both the value function and the optimal financial strategies. It turns out that in this case, as in [2,13], the convergence rate is super geometric, i.e. more rapid than any geometric one. The discovered "super geometric rate effect" significantly increases the speed of information processing and decision-making for practical problems of portfolio optimization in spread markets.

1.4. Organization of the paper

The rest of the paper is organized as follows. In Section 2 we introduce the Ornstein– Uhlenbeck financial market. The HJB equation and the optimal strategies are defined in Section 3. In Section 4 we describe the probability method for the analysis of the HJB equation. We state the main results of the paper in Section 5. Thereafter in Section 6, we study the properties of the FK mapping. In Section 7 we study the properties of the fixed point function. The stochastic optimal control method is given in Section 8. The Cauchy problem is stated in Section 9. The proofs of the main results are given in Section 10 and numeric simulations are presented in Section 11. Finally, in Section 12 we summarize our work. The auxiliary technic results are given in Appendix A.

2. Ornstein–Uhlenbeck model

Let $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0\le t\le T}, \mathbf{P})$ be a standard filtered probability space with $(\mathcal{F}_t)_{0\le t\le T}$ adapted Wiener processes $(W_t)_{0\le t\le T}$. Our financial market consists of one bond (riskless asset) $(\bar{S}_t)_{0\le t\le T}$ with the interest rate $r \ge 0$

$$\mathrm{d}\bar{S}_t = r\bar{S}_t \mathrm{d}t\,, \quad \bar{S}_0 = 1\,,$$

and *risky spread asset* $(S_t)_{0\le t\le T}$ defined by the stable Ornstein - Uhlenbeck process

$$\mathrm{d}S_t = -\kappa S_t \mathrm{d}t + \sigma \mathrm{d}W_t\,, \tag{2.1}$$

where the initial value S_0 is some fixed non random constant, $\kappa > 0$ is the market mean-reverting parameter from $\mathbb{R}$ and $\sigma > 0$ is the market volatility. We assume that the interest rate $0 \le r \le \kappa$. Let now $\bar{\alpha}_t$ and α_t be the numbers of the bonds and the risky assets at the moment $0 \le t \le T$ respectively. In this case the wealth process, $(X_t)_{0\le t\le T}$ is defined as

$$X_t = \bar{\alpha}_t \bar{S}_t + \alpha_t S_t\,.$$

We assume also that the consumption rate is given by a non-negative function $(c_t)_{0\le t\le T}$. Using the self financing principle with the consumption from [12] we get

$$\mathrm{d}X_t = \bar{\alpha}_t \mathrm{d}\bar{S}_t + \alpha_t \mathrm{d}S_t - \mathbf{c}_t \mathrm{d}t.$$

Replacing here the differentials $\mathrm{d}\bar{S}_t$ and $\mathrm{d}S_t$ by its definition in (2.1), we obtain the following stochastic differential equation for the wealth process

$$\mathrm{d}X_t = (rX_t - \kappa_1 \alpha_t S_t - \mathbf{c}_t)\mathrm{d}t + \alpha_t \sigma \mathrm{d}W_t\,, \tag{2.2}$$

where $\kappa_1 = \kappa + r$. Now we need to define all possible admissible strategies.

Definition 2.1. The financial strategy $\mathbf{u} = (\mathbf{u}_v)_{t\le v\le T}$ with $\mathbf{u}_v = (\alpha_v\,, \mathbf{c}_v)$ is called admissible on the time interval $[t, T]$ if it is $(\mathcal{F}_{t,v})_{t\le v\le T}$ progressively measurable stochastic process, where $\mathcal{F}_{t,v} = \sigma\{W_s - W_t,\, t \le s \le v\}$, such that

$$\int_t^T \alpha_v^2 \,\mathrm{d}v < \infty\,, \qquad \int_t^T \mathbf{c}_v \,\mathrm{d}v < \infty \qquad \text{a.s.} \tag{2.3}$$

and the equation (2.2) has a unique strong nonnegative solution. We denote by $\mathcal{A}_t$ the set of such admissible financial strategies.

For any $\mathbf{u} \in \mathcal{A}_t$ we introduce the objective function on the interval $[t, T]$ as

$$J(x, s, t; \mathbf{u}) := \mathbf{E}\left(\int_t^T \mathbf{c}_v^\gamma \mathrm{d}v + \beta X_T^\gamma | X_t = x,\, S_t = s \right), \tag{2.4}$$

where $0 < \gamma < 1$ and $\beta > 0$ are some fixed constants. Our goal is to maximize this objective function on the interval $[0, T]$, i.e.

$$\sup_{\mathbf{u}\in\mathcal{A}} J(x, s, 0; \mathbf{u})\,, \tag{2.5}$$

where $\mathcal{A} = \mathcal{A}_0$. To do this we use the dynamic programming method, according to which we need to study for any $0 \le t \le T$ the value function

$$J^*(x, s, t) = \sup_{\mathbf{u}\in\mathcal{A}_t} J(x, s, t; \mathbf{u})\,. \tag{2.6}$$

Remark 2.2. Note, that the problem (2.4) is optimal investment and consumption problem in classical setting for power utility functions (see, for example, [12] and the references therein). The coefficient $0 < \beta < \infty$ explains the investor preference between consumption and pure investment problem.

3. Hamilton–Jacobi–Bellman equation

In this section we introduce the Hamilton–Jacobi–Bellman (HJB) equation for the problem (2.6). Denoting by $\zeta_t = (X_t, S_t)$, we can rewrite equations (2.1) and (2.2) as,

$$\mathrm{d}\zeta_t = a(\zeta_t, \mathbf{u}_t)\mathrm{d}t + b(\zeta_t, \mathbf{u}_t)\mathrm{d}W_t, \tag{3.1}$$

where $\mathbf{u}_t = (\alpha_t, \mathbf{c}_t)$,

$$a(\zeta, u) = \begin{pmatrix} rx - \kappa_1 \alpha s - c \\ -\kappa s \end{pmatrix}, \quad b(\zeta, u) = \begin{pmatrix} \alpha\sigma \\ \sigma \end{pmatrix} \quad \text{and} \quad u = (\alpha, c)\,.$$

According to the dynamic programming method (see, for example, in [12, page 130]), for any $0 \le t \le T$ and any two times differentiable $\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ function z we set the Hamilton operator as

$$H(\zeta, t, \partial z, \partial^2 z) := \sup_{u \in \mathbb{R}\times\mathbb{R}_+} \left(a'(\zeta, t, u)\partial z + \frac{1}{2}\mathbf{tr}\left(bb'(\zeta, t, u)\partial^2 z\right) + c^\gamma \right), \tag{3.2}$$

where the prime $'$ denotes the transposition,

$$\partial z = \begin{pmatrix} z_x \\ z_s \end{pmatrix} \qquad \text{and} \qquad \partial^2 z = \begin{pmatrix} z_{xx} & z_{xs} \\ z_{sx} & z_{ss} \end{pmatrix}.$$

To construct optimal strategies for the problem (2.6), firstly one needs to study the HJB equation which is given by

$$\begin{cases} z_t(\zeta, t) + H(\zeta, t, \partial z, \partial^2 z) = 0, & t \in [0, T], \\ z(\zeta, T) = \beta x^\gamma, & \zeta \in \mathbb{R}_+ \times \mathbb{R}\,. \end{cases} \tag{3.3}$$

Note that, in this case the Hamilton function $H = +\infty$ if $z_{xx} \ge 0$ or $z_x \le 0$, and for $z_{xx} < 0$ and $z_x > 0$,

$$H(\zeta, t, \partial z, \partial^2 z) = a'(\zeta, t, u_0)\partial z + \frac{1}{2}\mathbf{tr}\left(bb'(\zeta, t, u_0)\partial^2 z\right) + c_0^\gamma\,,$$

where the optimal value $u_0 = (\alpha_0, c_0)$ is defined as

$$\alpha_0 = \frac{\kappa_1 s z_x}{\sigma^2 z_{xx}} - \frac{z_{sx}}{z_{xx}} \quad \text{and} \quad c_0 = \left(\frac{z_x}{\gamma}\right)^{\frac{1}{\gamma-1}}. \tag{3.4}$$

Using this in (3.3), we can represent the HJB equation as

$$\begin{cases} z_t + \dfrac{(\sigma^2 z_{xs} - \kappa_1 s z_x)^2}{2\sigma^2 |z_{xx}|} + \dfrac{\sigma^2 z_{ss}}{2} + rxz_x - \kappa s z_s + (1-\gamma)\left(\frac{z_x}{\gamma}\right)^{\frac{\gamma}{\gamma-1}} = 0\,, \\ z(\zeta, T) = \beta x^\gamma\,, \qquad \zeta \in \mathbb{R}_+ \times \mathbb{R}\,. \end{cases} \tag{3.5}$$

Using the separation variables method we will find a solution of this equation in the following exponential form

$$z(x,s,t) = \beta\, x^{\gamma}\, \exp\left\{\frac{s^2}{2} g(t) + Y(s,t)\right\}. \tag{3.6}$$

Here g is the solution of the ordinary differential equation

$$\dot{g}(t) - 2\gamma_2 g(t) + \gamma_1 g^2(t) + \gamma_3 = 0\,, \qquad g(T) = 0\,, \tag{3.7}$$

where the coefficients γ_i are defined as

$$\gamma_1 = \frac{\sigma^2}{1-\gamma}, \quad \gamma_2 = \frac{\gamma\kappa_1}{1-\gamma} + \kappa \quad \text{and} \quad \gamma_3 = \frac{\gamma\kappa_1^2}{(1-\gamma)\sigma^2}\,.$$

One check directly that if $0 \le r \le \kappa$, then the solution of (3.7) is represented as

$$g(t) = \vartheta_1 - \vartheta_2 - \frac{2\vartheta_2(\vartheta_1 - \vartheta_2)}{\vartheta_2 - \vartheta_1 + (\vartheta_1 + \vartheta_2)e^{2\vartheta_2\gamma_1(T-t)}} \ge 0\,, \tag{3.8}$$

where $\vartheta_1 = \gamma_2/\gamma_1$ and $\vartheta_2 = \sqrt{\vartheta_1^2 - \gamma_3/\gamma_1}$. Moreover, the function $Y = Y(s,t)$ in (3.6) is a solution of the non linear equation of parabolic type

$$\begin{cases} Y_t(s,t) + \dfrac{1}{2}\sigma^2 Y_{ss}(s,t) + s g_1(t) Y_s(s,t) + F\Big(s,t,Y,Y_s\Big) = 0\,, \\ Y(s,T) = 0\,, \end{cases} \tag{3.9}$$

where $g_1(t) = \gamma_1 g(t) - \gamma_2$ and F is the non negative function defined as

$$F(s,t,y,p) = \frac{\sigma^2 g(t)}{2} + r\gamma + \frac{\gamma_1 p^2}{2} + \beta_1\, G(s,t,y)\,, \quad \beta_1 = (1-\gamma)\beta^{\frac{1}{\gamma-1}} \tag{3.10}$$

and

$$G(s,t,y) = \exp\left\{-\frac{1}{1-\gamma}\left(\frac{s^2}{2}g(t) + y\right)\right\}.$$

To find optimal strategies for the problem (2.6) one needs to calculate the optimal control variables (3.4) for the HJB solution (3.6), i.e. we obtain that

$$\alpha_0(x,s,t) = \widetilde{\alpha}_0(s,t)x, \quad \text{and} \quad c_0(x,s,t) = \widetilde{c}_0(s,t)x\,.$$

Here the fractional coefficients are given as

$$\widetilde{\alpha}_0(s,t) = \frac{Y_s(s,t) - s g_2(t)\kappa_1/\sigma^2}{1-\gamma} \quad \text{and} \quad \widetilde{c}_0(s,t) = \beta^{\frac{1}{\gamma-1}} G(s,t,Y(s,t))\,, \tag{3.11}$$

where $g_2(t) = 1 - \sigma^2 g(t)/\kappa_1$. If we use in the equation (2.2) the strategy (3.11), then we obtain the following stochastic differential equation for the wealth process

$$\mathrm{d}X_t^* = a^*(t)X_t^*\mathrm{d}t + b^*(t)X_t^*\mathrm{d}W_t\,, \tag{3.12}$$

where

$$a^*(t) = r - \kappa_1 S_t \widetilde{\alpha}_0(S_t,t) - \widetilde{c}_0(S_t,t) \quad \text{and} \quad b^*(t) = \sigma \widetilde{\alpha}_0(S_t,t)\,.$$

Taking into account that the processes $a^*(t)$ and $b^*(t)$ are continuous, and using the Itô formula we obtain that

$$X^*_t = x\,\exp\left\{ \int_0^t b^*(v)\mathrm{d}W_v + \int_0^t \left(a^*(v) - \frac{1}{2}(b^*(v))^2 \right) \mathrm{d}v \right\}, \tag{3.13}$$

i.e. it is positive almost sure. Now, using this process, we set

$$\mathbf{u}^*_t = (\alpha^*_t, \mathbf{c}^*_t)\,, \quad \alpha^*_t = \widetilde{\alpha}_0(S_t,t)X^*_t \quad \text{and} \quad \mathbf{c}^*_t = \widetilde{c}_0(S_t,t)X^*_t\,. \tag{3.14}$$

Note that the processes $(\widetilde{\alpha}_0(S_t,t))_{0\le t\le T}$ and $\widetilde{c}_0(S_t,t))_{0\le t\le T}$ are the fractional investment and consumption strategies respectively. To show that the strategy (3.14) is optimal we will use the verification theorem method. To this end we need to impose some additional technical condition of Dirichlet type (see, for example, in [18]) for the optimal HJB process $z^*_t = z(X^*_t, S_t, t)$.

D) *For all $x \ge 0$, $s \in \mathbb{R}$ and $0 \le t \le T$ the family $\left(z^*_\tau\right)_{\tau\in\mathcal{M}_t}$ is uniformly integrable with respect to the conditional probability $\mathbf{P}\left(\cdot|X^*_t = x, S_t = s\right)$, where $\mathcal{M}_t$ is the set of all stopping times with the values in $[t,T]$.*

The class of these processes is called the Dirichlet Class (see, for example, in [18]). The condition **D**) plays a crucial role in the proof of the verification theorem.

Remark 3.1. Note that, the HJB solution for a pure investment problem given in [3] is obtained from (3.6) as

$$\lim_{\beta\to\infty} \frac{z(x,s,t)}{\beta} = x^\gamma\,\exp\left\{ \frac{s^2}{2} g(t) + Y(s,t) \right\} \tag{3.15}$$

with $Y(s,t) = (\sigma^2/2)\int_t^T g(u)\mathrm{d}u + \gamma r(T-t)$. One can check directly that the solution for pure investment problem can be obtained from (3.14) for $\beta \to \infty$ as

$$\alpha^*_t = -\frac{S_t g_2(t)}{1-\gamma} X^*_t\,, \quad \mathbf{c}_t = 0$$

and

$$\mathrm{d}X^*_t = \left(r + \kappa_1 \frac{S^2_t g_2(t)}{1-\gamma} \right) X^*_t \mathrm{d}t - \sigma\,\frac{S_t g_2(t)}{1-\gamma} X^*_t \mathrm{d}W_t\,, \quad X^*_0 = x\,.$$

4. Probability representation

In this paper we study the equation (3.9) on the basis of the probability representation method. To do this, first, for any $0 \le t \le T$ and $s \in \mathbb{R}$, we introduce the process $(\eta^{s,t}_u)_{t\le u\le T}$ as the solution of the following stochastic differential equation

$$\mathrm{d}\eta^{s,t}_u = g_1(u)\eta^{s,t}_u \mathrm{d}u + \sigma \mathrm{d}\widetilde{W}_u\,, \qquad \eta^{s,t}_t = s, \tag{4.1}$$

where $g_1(u)$ is defined in (3.9) and $(\widetilde{W}_u)_{u\ge 0}$ is a standard Brownian motion. It is clear that $\eta_u^{s,t} \sim \mathcal{N}(s\,\mu(u,t), \sigma_1^2(u,t))$, with

$$\mu(u,t) = e^{\int_t^u g_1(v)\mathrm{d}v} \quad \text{and} \quad \sigma_1^2(u,t) = \sigma^2 \int_t^u \mu^2(u,z)\mathrm{d}z\,. \tag{4.2}$$

To obtain the probability solution for (3.9) we set for any $h \in \mathbf{C}^{2,1}(\mathbb{R}\times[0,T])$

$$\mathcal{L}_h(s,t) = \int_t^T \mathbf{E}\,\Psi_h(\eta_u^{s,t},u)\mathrm{d}u\,, \quad \Psi_h(s,t) = F\Big(s,t,h(s,t),h_s(s,t)\Big) \tag{4.3}$$

and the function F is defined in (3.10). This operator is called the Feynman–Kac (FK) mapping (see, for example, in [2]). Note here that the function (3.8) is decreasing, i.e. $\max_{0\le t\le T} g(t) = g(0)$ and taking into account that for $0 \le r \le \kappa$

$$\vartheta_1^2 \ge \gamma_3/\gamma_1 + (1-\gamma)^2\kappa^2/\sigma^4\,,$$

we obtain that

$$g(0) < \vartheta_1 - \vartheta_2 \le \frac{\gamma\kappa_1^2}{\sigma^2(\gamma\kappa_1 + 2(1-\gamma)\kappa)} \le \frac{\gamma\kappa_1}{\sigma^2}\,. \tag{4.4}$$

Therefore, the function $g_1(t) = \gamma_1 g(t) - \gamma_2 \le \gamma_1 g(0) - \gamma_2 \le -\kappa < 0$ and

$$\sigma_1^2(u,t) \ge \sigma^2\mu^2(u,t)(u-t) \quad \text{for} \quad u > t\,. \tag{4.5}$$

This means that the Ornstein - Uhlenbeck process (4.1) is stable, i.e. converges weakly as $u\to\infty$. Now, we need to introduce some special metric space in which we will find the solution of the equation (3.9). To do this we denote by $\mathbf{C}_+^{1,0}\big(\mathbb{R}\times[0,T]\big)$ the set of positive functions from $\mathbf{C}^{1,0}\big(\mathbb{R}\times[0,T]\big)$, i.e. the set of continuous $\mathbb{R}\times[0,T]\to\mathbb{R}_+$ functions having the continuous partial derivatives in s. Now we study the cases for which the condition $\mathbf{D}$) holds true. To this end we impose the following condition.

$\mathbf{D}_1$) *The parameters in the objective function (2.4) are such that* $\beta \ge (16T/\pi)^{1-\gamma}$.

Using this condition we set

$$\mathcal{X} = \Big\{h \in \mathbf{C}_+^{1,0}\big(\mathbb{R}\times[0,T]\big): \quad \sup_{s,t} h(s,t) \le \mathbf{B}_0, \quad \sup_{s,t}|h_s(s,t)| \le \mathbf{B}_1\Big\}\,, \tag{4.6}$$

where $\mathbf{B}_0 = \big(\gamma\kappa_1/2 + r\gamma + \beta_1 + \gamma_1\mathbf{B}_1^2/2\big)\,T$ and

$$\mathbf{B}_1 = \frac{(1-\gamma)\sqrt{\pi}}{2\sigma\sqrt{2T}}\left(1 - \sqrt{\left(1 - \frac{16T}{\pi\beta^{\frac{1}{1-\gamma}}}\right)}\right)\,.$$

Now, for some $\nu > 1$ which will be precised later, we introduce the metric

$$\rho(f,h) = \sup_{s\in\mathbb{R}\,,0\le t\le T} e^{-\nu(T-t)}\Delta_{f,h}(s,t)\,, \tag{4.7}$$

where $\Delta_{f,h}(s,t) = |h(s,t) - f(s,t)| + |h_s(s,t) - f_s(s,t)|$.

Proposition 4.1. *The space* $(\mathcal{X}, \rho)$ *is a complete metric space.*

This proposition is shown in the same way of Proposition 5.1 in [2]. Now we can provide a probability solution for the equation (3.9).

Proposition 4.2. *The equation* (3.9) *has at least one bounded solution in* $\mathbf{C}^{2,1}(\mathbb{R} \times [0, T])$ *which coincides with the FK fixed point* h*, i.e.* $h = \mathcal{L}_h$*. Moreover, any FK fixed point* h *from* $\mathcal{X}$ *is a solution of the equation* (3.9)*.*

Proposition 4.3. *Assume that Condition* $\mathbf{D}_1$*) holds true. Then for the wealth process* (3.12) *defined through the FK fixed point* h *from* $\mathcal{X}$ *Condition* $\mathbf{D}$*) is also satisfied.*

These propositions are shown in Appendices A.1 and A.2 respectively.

Remark 4.4. It should be noted, that we can't apply the usual methods to show the uniqueness of the classical solutions for the equation (3.9) since the coefficient of the partial derivative Y_s is not bounded in $\mathbb{R}$ (see, for example, Theorem 8.1 on the p. 495 in [14]).

5. Main results

First of all we study the strategy (3.14).

Theorem 5.1. *Assume that the condition* $\mathbf{D}$) *holds, then the equation* (3.9) *has the unique bounded solution and the value function (2.6) is given by*

$$U(x, s, t) = J(x, s, t, \mathbf{u}^*) = \beta x^\gamma \exp\left\{\frac{s^2}{2} g(t) + Y(s, t)\right\}, \tag{5.1}$$

where the optimal strategy $\mathbf{u}^* = (\alpha_s^*, c_s^*)_{t \le s \le T}$ *is defined in* (3.14)*.*

Remark 5.2. It should be noted that Proposition 4.2 and Theorem 5.1 hold true for the more general condition when $0 \le r \le \kappa/\sqrt{\gamma}$ which provides the representation of the function g in the form (3.8). In this paper we assume that $0 \le r \le \kappa$ to obtain the upper bound (4.4) which plays the key role to check the condition $\mathbf{D}$). Indeed, it is not necessary to consider the case in which the interest rate r is large. We can always to reduce this rate including the corresponding part in the consumption for the problem (2.5). In practice the interest rate r is sufficiently small, contrary to the coefficient κ which cannot be small, since for small coefficient the model (2.1) tends to the Brownian motion which cannot be a good model for the spread markets based on the difference of the co-integrated assets. So, the condition that $0 \le r \le \kappa$ is very natural for the model (2.1).

Now we have to study the FK fixed point function h. To this end we define the iterative sequence $(h_n)_{n \ge 1}$ as

$$h_n = \mathcal{L}_{h_{n-1}}, \quad n \ge 1 \tag{5.2}$$

and h_0 is an arbitrary function from $\mathcal{X}$, for example, $h \equiv 0$.

Theorem 5.3. *Under Condition* $\mathbf{D}_1$*) the solution of the equation* (3.9) *coincides with the FK fixed point function from* $\mathcal{X}$*. Moreover, the sequence* (5.2) *goes to this function such that for any* $0 < \delta < 1/2$

$$\lim_{n\to\infty} n^{\delta n} \parallel h - h_n \parallel = 0\,, \tag{5.3}$$

where $\parallel f \parallel = \sup_{s\in\mathbb{R}} \sup_{0\le t\le T} (|f(s,t)| + |f_s(s,t)|)$.

To calculate the optimal strategies, we use the approximations (5.2) in the fractional strategies (3.11), i.e. we set

$$\widetilde{\alpha}_{0,n}(s,t) = \frac{1}{\sigma^2(1-\gamma)}\left(\sigma^2 s g(t) + \sigma^2 \frac{\partial}{\partial s} h_n(s,t) - \kappa_1 s\right)$$

and

$$\widetilde{c}_{0,n}(s,t) = \beta^{\frac{1}{\gamma-1}} G(s,t,h_n(s,t))\,.$$

Theorem 5.3 and the definitions of α_0 and c_0 in (3.14) imply the following result.

Theorem 5.4. *Under Condition* $\mathbf{D}_1$) *for any* $0 < \delta < 1/2$,

$$\lim_{n\to\infty} n^{\delta n} \sup_{s\in\mathbb{R}} \sup_{0\le t\le T} \left(\left|\widetilde{\alpha}_0(s,t) - \widetilde{\alpha}_{0,n}(s,t)\right| + \left|\widetilde{c}_0(s,t) - \widetilde{c}_{0,n}(s,t)\right|\right) = 0\,.$$

Now using these approximations we set

$$\alpha^n_t = \widetilde{\alpha}_{0,n}(S_t,t) X^n_t \quad \text{and} \quad c^n_t = \widetilde{c}_{0,n}(S_t,t) X^n_t\,. \tag{5.4}$$

Here

$$\mathrm{d}X^n_t = a^*_n(t) X^n_t \mathrm{d}t + b^*_n(t) X^n_t \mathrm{d}W_t\,, \quad X^n_0 = x\,,$$

where

$$a^*_n(t) = r - \kappa_1 S_t \widetilde{\alpha}_{0,n}(S_t,t) - \widetilde{c}_{0,n}(S_t,t) \quad \text{and} \quad b^*_n(t) = \sigma \widetilde{\alpha}_{0,n}(S_t,t)\,.$$

Finally, we can obtain the approximations for the optimal strategies (3.14).

Theorem 5.5. *Under Condition* $\mathbf{D}_1$) *for any* $0 < \delta < 1/2$,

$$\mathbf{P} - \lim_{n\to\infty} n^{\delta n} \sup_{0\le t\le T} \left(\left|\alpha^n_t - \alpha^*_t\right| + \left|c^n_t - c^*_t\right|\right) = 0\,. \tag{5.5}$$

Theorems 5.1, 5.3 and 5.5 are shown in Section 10.

Remark 5.6. Note that, similarly to [2] the convergence rate for the iterative scheme is super geometric, i.e. more rapid than any geometric ones.

Remark 5.7. Note that the condition $\mathbf{D}_1$) is nor restricted. Indeed, if $\beta < (16T/\pi)^{1-\gamma}$, i.e. $T > \beta^{1/(1-\gamma)}\pi/16 := T_*$, then one needs to divide the interval $[0,T]$ in m parts $0 = t_0 < t_1 ... < t_m = T$ with $\max_{1\le j\le m}(t_j - t_{j-1}) \le T_*$ and to use the strategy (3.14) on each interval $[t_{j-1}, t_j]$.

6. Properties of the FK mapping

Now we study the main properties of the mapping (4.3).

Proposition 6.1. *Under Condition* $\mathbf{D}_1$) *the FK mapping is* $\mathcal{X}$ *closed, i.e.* $\mathcal{L}_h : \mathcal{X} \to \mathcal{X}$.

Proof. First, note that we can represent the function $\mathcal{L}_h(s,t)$ as

$$\mathcal{L}_h(s,t) = \frac{\sigma^2}{2} \int_t^T g(u)\mathrm{d}u + r\gamma(T-t) + \frac{\gamma_1}{2}\mathbf{E} \int_t^T h_s^2(\eta_u^{s,t}, u)\mathrm{d}u$$
$$+ \beta_1 \mathbf{E} \int_t^T G(\eta_u^{s,t}, u, h(\eta_u^{s,t}, u))\mathrm{d}u\,.$$

Using the inequality (4.4) and the bounds $\mathbf{B}_0$ and $\mathbf{B}_1$ defined in (4.6), we obtain

$$\mathcal{L}_h(s,t) \le \frac{\sigma^2}{2} g(0)(T-t) + \frac{\gamma_1}{2}\mathbf{B}_1^2(T-t) + r\gamma(T-t) + \beta_1(T-t) \le \mathbf{B}_0\,. \tag{6.1}$$

Then by taking the derivative in s, we get

$$\frac{\partial}{\partial s}\mathcal{L}_h(s,t) = \frac{\gamma_1}{2}\frac{\partial}{\partial s}\mathbf{E}\int_t^T h_s^2(\eta_u^{s,t}, u)\mathrm{d}u + \beta_1 \frac{\partial}{\partial s}\mathbf{E}\int_t^T G(\eta_u^{s,t}, u, h(\eta_u^{s,t}, u))\mathrm{d}u.$$

From Lemma A.2 in Appendix A.5 and as $0 < G(s,t,y) \le 1$, we have

$$\left|\frac{\partial}{\partial s}\mathcal{L}_h(s,t)\right| \le \frac{\gamma_1}{\sigma}\sqrt{\frac{2(T-t)}{\pi}}\mathbf{B}_1^2 + \beta_1\frac{2}{\sigma}\sqrt{\frac{2(T-t)}{\pi}}.$$

Taking into account the definition $\mathbf{B}_1$ in (4.6) we obtain,

$$\left|\frac{\partial}{\partial s}\mathcal{L}_h(s,t)\right| \le \frac{\gamma_1}{\sigma}\sqrt{\frac{2T}{\pi}}\mathbf{B}_1^2 + \beta_1\frac{2}{\sigma}\sqrt{\frac{2T}{\pi}} = \mathbf{B}_1\,.$$

So, we get that $\mathcal{L}_h \in \mathcal{X}$. Hence Proposition 6.1. □

Proposition 6.2. *For all* $f \in \mathcal{X}$, *for all* $s \in \mathbb{R}$ *and* $0 \le t \le T$,

$$\frac{\partial}{\partial s}\mathcal{L}_f(s,t) = \int_t^T \left(\int_{\mathbb{R}} F\Big(z, t, f(z,u), f_s(z,u)\Big)\, \varphi_1(s,t,z,u)\mathrm{d}z\right)\mathrm{d}u\,.$$

Here the function F *is defined in* (3.10) *and*

$$\varphi_1(s,t,z,u) = \frac{\partial}{\partial s}\varphi(s,t,z,u) = \mathbf{v}\frac{\mu(u,t)}{\sigma_1(u,t)}\, \varphi(s,t,z,u)\,, \tag{6.2}$$

where

$$\varphi(s,t,z,u) = \frac{e^{-\frac{\mathbf{v}^2}{2}}}{\sqrt{2\pi}\sigma_1(u,t)} \quad \text{and} \quad \mathbf{v} = \mathbf{v}(s,t,z,u) = \frac{z - s\mu(u,t)}{\sigma_1(u,t)}\,.$$

The proof is given in Appendix A.3. Now we set

$$\mathbf{r}_* = \max\left(1, \left(1+\sqrt{2}/\sigma\right)\left(\gamma_1 \mathbf{B}_1 + \beta^{\frac{1}{\gamma-1}}\right)\right), \tag{6.3}$$

where the coefficient $\mathbf{B}_1$ is given in (4.6).

Proposition 6.3. *For any $\nu > \mathbf{r}_*^2$ in the metric (4.7) the mapping $\mathcal{L}$ is contraction in $\mathcal{X}$, i.e. for any $h, f \in \mathcal{X}$,*

$$\rho(\mathcal{L}_h, \mathcal{L}_f) \leq \lambda \rho(h, f) \quad \text{and} \quad \lambda = \frac{\mathbf{r}_*}{\sqrt{\nu}} < 1\,. \tag{6.4}$$

Proof. Using the definition (4.3), we obtain that for any h and f from $\mathcal{X}$,

$$\begin{aligned}\mathcal{L}_h - \mathcal{L}_f =& \frac{\gamma_1}{2}\mathbf{E}\int_t^T \left(h_s^2(\eta_u^{s,t}, u) - f_s^2(\eta_u^{s,t}, u)\right)\mathrm{d}u \\ &+ \beta_1 \mathbf{E}\int_t^T \left(G\left(\eta_u^{s,t}, u, h(\eta_u^{s,t}, u)\right) - G\left(\eta_u^{s,t}, u, f(\eta_u^{s,t}, u)\right)\right)\mathrm{d}u\,.\end{aligned}$$

Taking into account that the function G defined in (3.10) is lipschitzian, i.e. for any $y_1 \geq 0$ and $y_2 \geq 0$

$$\left|G(s,t,y_1) - G(s,t,y_2)\right| \leq \frac{1}{1-\gamma}|y_1 - y_2|\,,$$

we obtain that

$$\begin{aligned}|\mathcal{L}_h - \mathcal{L}_f| \leq& \frac{\gamma_1}{2}\left|\int_t^T \mathbf{E}\left(h_s^2(\eta_u^{s,t}, u) - f_s^2(\eta_u^{s,t}, u)\right)\mathrm{d}u\right| \\ &+ \beta^{\frac{1}{\gamma-1}}\mathbf{E}\int_t^T \left|h(\eta_s^{s,t}, u) - f(\eta_u^{s,t}, u)\right|\mathrm{d}u.\end{aligned} \tag{6.5}$$

Recall that f and h belong to $\mathcal{X}$, i.e. the difference for the squares of their derivatives can be estimated as $\left|h_s^2(z,u) - f_s^2(z,u)\right| \leq 2\mathbf{B}_1|h_s(z,u) - f_s(z,u)|$. Therefore,

$$\left|\mathcal{L}_h(s,t) - \mathcal{L}_f(s,t)\right| \leq \mathbf{B}_2 \int_t^T \Delta_{h,f}^*(u)\, e^{-\nu(T-u)}\, e^{\nu(T-u)}\mathrm{d}u,$$

where $\mathbf{B}_2 = \gamma_1\mathbf{B}_1 + \beta^{\frac{1}{\gamma-1}}$ and $\Delta_{h,f}^*(t) = \sup_{y\in\mathbb{R}} \Delta_{h,f}(y,t)$. Using here (4.7) we get

$$\left|\mathcal{L}_h(s,t) - \mathcal{L}_f(s,t)\right| \leq \mathbf{B}_2 \frac{\rho(h,f)}{\nu}\, e^{\nu(T-t)}\,.$$

Moreover, using Proposition 6.2, we obtain that

$$\left|\frac{\partial}{\partial s}\mathcal{L}_h(s,t) - \frac{\partial}{\partial s}\mathcal{L}_f(s,t)\right| = \left|\int_t^T \int_{\mathbb{R}} \frac{\gamma_1}{2}\left(h_s^2(z,u) - f_s^2(z,u)\right)\right.$$

$$+ \beta_1(G(z,u,h(z,u)) - G(z,u,f(z,u))\Big)\varphi_1(s,t,z,u)\mathrm{d}z\mathrm{d}u\Big| .$$

Now note that in view of the bound (4.5), for $u > t$

$$\sup_{s\in\mathbb{R}} \int_{\mathbb{R}} |\varphi_1(s,t,z,u)|\mathrm{d}z \le \frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{u-t}} . \tag{6.6}$$

It should be noted also that

$$\left|\frac{\gamma_1}{2}(h_s^2(z,u) - f_s^2(z,u)) + \beta_1(G(z,u,h(z,u)) - G(z,u,f(z,u))\right| \le \mathbf{B}_2\Delta^*_{f,h}(u) .$$

Thus, the bound (6.6) implies

$$\begin{aligned}\left|\frac{\partial}{\partial s}\mathcal{L}_h(s,t) - \frac{\partial}{\partial s}\mathcal{L}_f(s,t)\right| &\le \mathbf{B}_2 \int_t^T \Delta^*_{f,h}(u) \int_{\mathbb{R}} |\varphi_1(s,t,z,u)|\mathrm{d}z\,\mathrm{d}u \\ &\le \frac{\mathbf{B}_2}{\sigma}\sqrt{\frac{2}{\pi}} \int_t^T \frac{\Delta^*_{f,h}(u)}{\sqrt{u-t}}\,\mathrm{d}u .\end{aligned}$$

Using again here the definition (4.7) and the fact that $\sqrt{\pi} = \int_0^{+\infty} e^{-z}z^{-1/2}\mathrm{d}z$, we get

$$\begin{aligned}\Big|\frac{\partial}{\partial s}\mathcal{L}_h(s,t) - \frac{\partial}{\partial s}\mathcal{L}_f(s,t)\Big| &\le \frac{\mathbf{B}_2}{\sigma}\sqrt{\frac{2}{\pi}}\,\rho(f,h)\int_t^T \frac{e^{\nu(T-u)}}{\sqrt{u-t}}\mathrm{d}u \\ &\le \frac{\mathbf{B}_2}{\sigma}\sqrt{\frac{2}{\pi}}\,\rho(f,h)\,e^{\nu(T-t)}\int_t^T \frac{e^{-\nu(u-t)}}{\sqrt{u-t}}\mathrm{d}u \le \frac{\sqrt{2}\mathbf{B}_2}{\sigma}\,\rho(f,h)\frac{e^{\nu(T-t)}}{\sqrt{\nu}} .\end{aligned}$$

Therefore, for any $\nu > 1$ we get $\rho(\mathcal{L}_h,\mathcal{L}_f) \le (\mathbf{r}_*/\sqrt{\nu})\rho(f,h)$ where the coefficient $\mathbf{r}_*$ is defined in (6.3). Hence Proposition 6.3. □

Proposition 6.4. *Assume that $\beta \ge (16T/\pi)^{1-\gamma}$. Then for the mapping $\mathcal{L}$ there exists a unique fixed point h in $\mathcal{X}$, i.e. $\mathcal{L}_h = h$. Moreover, for any $n \ge 1$ and $\nu > \mathbf{r}_*^2$ the approximation sequence (5.2) satisfies the following inequality*

$$\rho(h,h_n) \le \frac{2(\mathbf{B}_0 + \mathbf{B}_1)}{1-\lambda}\lambda^n \quad \text{and} \quad \lambda = \frac{\mathbf{r}_*}{\sqrt{\nu}} < 1 , \tag{6.7}$$

where the coefficients $\mathbf{B}_0$, $\mathbf{B}_1$ and $\mathbf{r}_*$ are given in (4.6) and in (6.3) respectively.

Proof. Indeed, Proposition 6.3 implies $\rho(h_n,h_{n+1}) = \rho(\mathcal{L}_{h_{n-1}},\mathcal{L}_{h_n}) \le \lambda\rho(h_{n-1},h_n)$. Therefore,

$$\rho(h_n,h_{n+1}) \le \lambda\rho(\mathcal{L}_{h_{n-1}},\mathcal{L}_{h_n}) \le \lambda^2\rho(h_{n-2},h_{n-1}) \le \ldots \le \lambda^n\rho(h_0,h_1).$$

Note that from the definitions (4.6) and (4.7) we can obtain directly that for any h_0 and h_1 from $\mathcal{X}$ the metric $\rho(h_0,h_1) \le 2(\mathbf{B}_0 + \mathbf{B}_1)$. So, for $m > n$,

$$\rho(h_n, h_m) \le 2(\mathbf{B}_0 + \mathbf{B}_1)(\lambda^n + \lambda^{n+1} + \ldots + \lambda^{m-1}) \le 2(\mathbf{B}_0 + \mathbf{B}_1)\sum_{i=n}^{\infty} \lambda^i .$$

Therefore, in view of Proposition 4.1, there exists a unique fixed point $h \in \mathcal{X}$ for the mapping $\mathcal{L}$ which satisfies the inequality (6.7). Hence Proposition 6.4. □

7. Properties of the FK fixed point function

In this section we study some regularity properties for the function h. First we study the smoothness with respect to the variable s.

Proposition 7.1. *If $h \in \mathcal{X}$ is a FK fixed point, i.e. $h = \mathcal{L}_h$, then for any $0 < \varepsilon < 1$,*

$$\sup_{0\le t\le T} \sup_{s_1,s_2\in\mathbb{R}} \frac{|h_s(s_1,t) - h_s(s_2,t)|}{|s_1 - s_2|^{\varepsilon}} < \infty.$$

Proof. In view of Proposition 6.2 and the definition (4.3), the partial derivative of h can be represented as

$$\frac{\partial}{\partial s} h(s,t) = \int_t^T \int_{\mathbb{R}} \Psi_h(z,u)\, \varphi_1(s,t,z,u) \mathrm{d}z \mathrm{d}u \,,$$

i.e.

$$\left| \frac{\partial}{\partial s} h(s_1,t) - \frac{\partial}{\partial s} h(s_2,t) \right| \le \int_t^T \int_{\mathbb{R}} |\Psi_h(z,u)|\, |\varphi_1(s_1,t,z,u) - \varphi_1(s_2,t,z,u)|\, \mathrm{d}z \mathrm{d}u \,.$$

Note that for any h from $\mathcal{X}$ the function $\Psi_h(z,u)$ is bounded, i.e.

$$\Psi^* = \sup_{h\in\mathcal{X}} \sup_{z\in\mathbb{R}\,,0\le u\le T} |\Psi_h(z,u)| < \infty \,. \tag{7.1}$$

Therefore, if $\Delta = |s_1 - s_2| \ge 1$ then, using the bound (6.6), we get

$$\begin{aligned} \frac{1}{\Delta^{\varepsilon}} \left| \frac{\partial}{\partial s} h(s_1,t) - \frac{\partial}{\partial s} h(s_2,t) \right| &\le \Psi^* \int_t^T \int_{\mathbb{R}} |\varphi_1(s_1,t,z,u)| \mathrm{d}z \, \mathrm{d}u \\ &+ \Psi^* \int_t^T \int_{\mathbb{R}} |\varphi_1(s_2,t,z,u)| \mathrm{d}z \, \mathrm{d}u \, < \infty \,. \end{aligned}$$

Let now $0 < \Delta < 1$. Then,

$$\frac{1}{\Delta^{\varepsilon}} \left| \frac{\partial}{\partial s} h(s_1,t) - \frac{\partial}{\partial s} h(s_2,t) \right| \le \Psi^* \, I(\Delta) \,, \tag{7.2}$$

where

$$I(\Delta) = \int_t^T \int_{\mathbb{R}} \frac{|\varphi_1(s_1,t,z,u) - \varphi_1(s_2,t,z,u)|}{\Delta^{\varepsilon}} \mathrm{d}z \, \mathrm{d}u \,.$$

Then for $\Delta_1 = \Delta^{2\varepsilon}$ we can rewrite it as

$$I(\Delta) = \int_t^{t_1} \int_{\mathbb{R}} \frac{\left|\varphi_1(s_1,t,z,u) - \varphi_1(s_2,t,z,u)\right|}{\Delta^\varepsilon} \mathrm{d}z\,\mathrm{d}u + \int_{t_1}^{T} \int_{\mathbb{R}} \frac{\left|\varphi_1(s_1,t,z,u) - \varphi_1(s_2,t,z,u)\right|}{\Delta^\varepsilon} \mathrm{d}z\,\mathrm{d}u\,,$$

where $t_1 = t + \Delta_1$. So,

$$I(\Delta) \le \frac{1}{\Delta^\varepsilon} \int_t^{t_1} \left(\int_{\mathbb{R}} \left|\varphi_1(s_1,t,z,u)\right| \mathrm{d}z + \int_{\mathbb{R}} \left|\varphi_1(s_2,t,z,u)\right| \mathrm{d}z \right) \mathrm{d}u + \frac{1}{\Delta^\varepsilon} \int_{t_1}^{T} \int_{\mathbb{R}} \left|\varphi_1(s_1,t,z,u) - \varphi_1(s_2,t,z,u)\right| \mathrm{d}z\mathrm{d}u.$$

Taking into account again the bound (6.6), we estimate the integral $I(\Delta)$ as

$$I(\Delta) \le \frac{4}{\sigma}\sqrt{\frac{2}{\pi}} + \frac{1}{\Delta^\varepsilon} \int_{t_1}^{T} \int_{\mathbb{R}} \left|\varphi_1(s_1,t,z,u) - \varphi_1(s_2,t,z,u)\right| \mathrm{d}z\,\mathrm{d}u.$$

Then

$$I(\Delta) \le \frac{4}{\sigma}\sqrt{\frac{2}{\pi}} + \frac{1}{\Delta^\varepsilon} \int_{t_1}^{T} \int_{s_1}^{s_2} \int_{\mathbb{R}} \left|\varphi_2(s,t,z,u)\right| \mathrm{d}z\,\mathrm{d}s\,\mathrm{d}u,$$

where

$$\varphi_2 = \frac{\partial}{\partial s}\varphi_1(s,t,z,u) = \frac{\mu^2}{\sqrt{2\pi}\sigma_1^3}\, e^{\frac{-\mathbf{v}^2}{2}} \left(\mathbf{v}^2 - 1\right) \quad \text{and} \quad \mathbf{v} = \frac{z - s\mu(u,t)}{\sigma_1(u,t)}\,.$$

Thus, on view of the lower bound (4.5)

$$\int_{\mathbb{R}} |\varphi_2(s,t,z,u)| \mathrm{d}z \le \frac{\mu^2}{\sqrt{2\pi}\sigma_1^2} \int_{\mathbb{R}} (\mathbf{v}^2 + 1)\, e^{\frac{-\mathbf{v}^2}{2}} \mathrm{d}\mathbf{v} \le \frac{2}{\sigma^2(u-t)}$$

and, therefore,

$$I(\Delta) \le \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} + \frac{2}{\sigma^2}\Delta^{1-\varepsilon} \int_{t_1}^{T} \frac{1}{u-t}\,\mathrm{d}u \le \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} + \frac{2}{\sigma^2}\,\Delta^{1-\varepsilon}(|\ln \Delta_1| + |\ln T|)\,.$$

Hence Proposition 7.1. □

Now, we study the smoothness properties for the function h with respect to t.

Proposition 7.2. *Let $h = \mathcal{L}_h$, with $h \in \mathcal{X}$. Then, for all $N \geq 1$ and $0 < \varepsilon < 1/2$,*

$$\sup_{0\le t_1<t_2\le T} \sup_{|s|\le N} \frac{\left|h(s,t_2)-h(s,t_1)\right| + \left|h_s(s,t_2)-h_s(s,t_1)\right|}{(t_2-t_1)^{\varepsilon}} < \infty\,.$$

Proof. Firstly, note that

$$h(s,t) = \int_t^T \overline{\Psi}_h(s,t,u)\mathrm{d}u \quad \text{and} \quad \overline{\Psi}_h(s,t,u) = \int_{\mathbb{R}} \Psi_h(z,u)\varphi(s,t,z,u)\,\mathrm{d}z.$$

Therefore, for any $0 \le t_1 < t_2 \le T$

$$h(s,t_2)-h(s,t_1) = \int_{t_2}^T \Big(\overline{\Psi}_h(s,t_2,u)\mathrm{d}u - \overline{\Psi}_h(s,t_1,u)\Big)\mathrm{d}u - \int_{t_1}^{t_2} \overline{\Psi}_h(s,t_1,u)\mathrm{d}u\,.$$

Let now $\Delta = t_2 - t_1$ and $\Delta_1 = \Delta^{2\varepsilon}$ for some $0 < \varepsilon < 1/2$. Taking into account that the function $\overline{\Psi}_h(s,t,u)$ is bounded, we obtain that for some $0 < \mathbf{c}^* < \infty$

$$\frac{1}{\Delta^{\varepsilon}}\left|h(s,t_2)-h(s,t_1)\right| \le \mathbf{c}^* \left(\frac{1}{\Delta^{\varepsilon}} I(\Delta) + \Delta^{1-\varepsilon}\right)\,, \tag{7.3}$$

where $I(\Delta) = \int_{t_2}^T \int_{\mathbb{R}} \overline{\varphi}(z,u)\mathrm{d}z\mathrm{d}u$ and $\overline{\varphi}(z,u) = |\varphi(s,t_2,z,u) - \varphi(s,t_1,z,u)|$. We represent this term as $I(\Delta) = I_1(\Delta) + I_2(\Delta)$, where

$$I_1(\Delta) = \int_{t_2}^{t_2+\Delta_1} \int_{\mathbb{R}} \overline{\varphi}(z,u)\mathrm{d}z\mathrm{d}u \quad \text{and} \quad I_2(\Delta) = \int_{t_2+\Delta_1}^{T} \int_{\mathbb{R}} \overline{\varphi}(z,u)\mathrm{d}z\mathrm{d}u.$$

Since $\int_{\mathbb{R}} \overline{\varphi}(z,u)\mathrm{d}z \le 2$, we get $I_1(\Delta) \le 2\Delta_1$. To estimate $I_2(\Delta)$ note, that

$$\overline{\varphi}(z,u) = |\varphi(s,t_2,z,u) - \varphi(s,t_1,z,u)| \le \int_{t_1}^{t_2} \left|\varphi_t(s,\theta,z,u)\right|\mathrm{d}\theta\,,$$

where

$$\varphi_t(s,t,z,u) = \frac{\partial}{\partial t}\varphi(s,t,z,u) = -\left(\frac{\sigma_2}{2\,\sqrt{2\pi}\,\sigma_1^3} + \frac{\mathbf{v}\dot{\mathbf{v}}}{\sqrt{2\pi}\sigma_1}\right) e^{-\frac{\mathbf{v}^2}{2}}\,,$$

the dote $\dot{}$ is the derivative with respect to t, $\sigma_2 = \dot{\mathbf{s}}$ and $\mathbf{s} = \sigma_1^2$. Note, here that

$$\dot{\mathbf{v}} = -\frac{s\dot{\mu}}{\sigma_1} - \frac{z - s\mu}{\sigma_1^2}\dot{\sigma}_1 = -\frac{s\dot{\mu}}{\sigma_1} - \frac{\mathbf{v}\sigma_2}{2\sigma_1^2}\,.$$

Moreover, using the inequality (4.5) and taking into account, that $\dot{\mu}$ and σ_2 are bounded, we obtain that for some $\mathbf{c}^* > 0$ and $u > t$

$$\left|\frac{\partial}{\partial t}\varphi(s,t,z,u)\right| \le \mathbf{c}^* (1+|s|)\,\frac{(\mathbf{v}^2+|\mathbf{v}|+1)}{\sigma_1^3}\,e^{-\frac{\mathbf{v}^2}{2}}$$
$$\le \mathbf{c}^* (1+|s|)\,\frac{(\mathbf{v}^2+|\mathbf{v}|+1)}{\sigma_1(u-t)}\,e^{-\frac{\mathbf{v}^2}{2}}\,.$$

Therefore, for some $\mathbf{c}^* > 0$

$$\int_{\mathbb{R}} \left|\frac{\partial}{\partial t}\varphi(s,t,z,u)\right| \mathrm{d}z \le \frac{\mathbf{c}^*(1+|s|)}{u-t},$$

and we get

$$|I_2(\Delta)| \le \mathbf{c}^*\,(1+|s|) \int_{t_1}^{t_2} \left(\int_{t_2+\Delta_1}^{T} \frac{1}{u-\theta}\mathrm{d}u \right) \mathrm{d}\theta$$
$$\le \mathbf{c}^*\,(1+|s|)\,\Delta \int_{t_2+\Delta_1}^{T} \frac{\mathrm{d}u}{u-t_2} \le \mathbf{c}^*\,(1+|s|)\Delta|\ln\Delta_1|.$$

Therefore,

$$\limsup_{\Delta\to 0} \frac{1}{\Delta^{\varepsilon}} \sup_{s\in\mathbb{R}} \frac{\left|h(s,t_2)-h(s,t_1)\right|}{1+|s|} < \infty\,.$$

Now through Proposition 6.2 we obtain that

$$\frac{\partial}{\partial s}h(s,t) = \frac{\partial}{\partial s}\mathcal{L}_h(s,t) = \frac{1}{\sqrt{2\pi}} \int_t^T \frac{\mu(u,t)}{\sigma_1^2(u,t)} \left(\int_{\mathbb{R}} \Psi_h(z,u)\mathbf{v}\ e^{-\frac{\mathbf{v}^2}{2}}\,\mathrm{d}z \right) \mathrm{d}u\,, \tag{7.4}$$

i.e. this derivative can be represented as

$$\frac{\partial}{\partial s}h(s,t) = \int_t^T \frac{\mu(u,t)}{\sigma_1(u,t)} \left(\int_{\mathbb{R}} \Psi_h(s\mu+\sigma_1\mathbf{v},u)\mathbf{v}\ \frac{e^{-\frac{\mathbf{v}^2}{2}}}{\sqrt{2\pi}}\mathrm{d}\mathbf{v} \right) \mathrm{d}u$$
$$= \int_t^T \frac{\mu(u,t)}{\sigma_1(u,t)} \left(\mathbf{E}\ \Psi_h(s\mu(u,t)+\sigma_1(u,t)\ \xi,u)\xi \right) \mathrm{d}u$$
$$= \int_t^T \mathbf{q}_1(t,u)\mathbf{q}_2(t,u)\mathrm{d}u\,,$$

where $\xi \sim \mathcal{N}(0,1)$, $\mathbf{q}_1(t,u) = \mathbf{E}\xi\Psi_h(s\mu(u,t)+\sigma_1(u,t)\xi,u)$ and $\mathbf{q}_2(t,u) = \mu(u,t)/\sigma_1(u,t)$. Setting now $\mathbf{q}_3(u) = \mathbf{q}_1(t_2,u)\mathbf{q}_2(t_2,u) - \mathbf{q}_1(t_1,u)\mathbf{q}_2(t_1,u)$, we obtain that

$$\frac{\partial}{\partial s}h(s,t_2) - \frac{\partial}{\partial s}h(s,t_1) = \int_{t_2}^{T} \mathbf{q}_3(u)\mathrm{d}u - \int_{t_1}^{t_2} \mathbf{q}_1(t_1,u)\mathbf{q}_2(t_1,u)\mathrm{d}u\,.$$

Moreover, note that the inequality (4.5) implies, that for $u > t$,

$$\mathbf{q}_2(u,t) = \frac{\mu(u,t)}{\sigma_1(u,t)} \le \frac{1}{\sigma\sqrt{u-t}}\,. \tag{7.5}$$

Now we recall, that the function Ψ_h is bounded. Therefore, we get that for some $\mathbf{c}^* > 0$

$$\left|\frac{\partial}{\partial s}h(s,t_2) - \frac{\partial}{\partial s}h(s,t_1)\right| \le \int_{t_2}^{T} \left|\mathbf{q}_3(u)\right| \mathrm{d}u + \int_{t_1}^{t_2} \frac{\mathbf{c}^*}{\sqrt{u-t_1}} \mathrm{d}u \le I_1^*(\Delta) + I_2^*(\Delta) + 2\mathbf{c}^*\sqrt{\Delta}\,,$$

where $I_1^*(\Delta) = \int_{t_2}^{t_2+\Delta_1} \left|\mathbf{q}_3(u)\right| \mathrm{d}u$ and $I_2^*(\Delta) = \int_{t_2+\Delta_1}^{T} \left|\mathbf{q}_3(u)\right| \mathrm{d}u$. First note, that

$$I_1^*(\Delta) \le \mathbf{c}^* \int_{t_2}^{t_2+\Delta_1} \left(\frac{1}{\sqrt{u-t_2}} + \frac{1}{\sqrt{u-t_1}}\right) \mathrm{d}u \le 4\mathbf{c}^*\sqrt{\Delta_1}.$$

To study $I_2^*(\Delta)$ we use the bound (7.5) which implies that for any $u > t_2$

$$\left|\mathbf{q}_3(u)\right| \le \mathbf{c}^* \left(\left|\mathbf{q}_2(u,t_2) - \mathbf{q}_2(u,t_1)\right| + \frac{1}{\sqrt{u-t_1}} \left|\mathbf{q}_1(u,t_2) - \mathbf{q}_1(u,t_1)\right| \right).$$

From the definition of $\mathbf{q}_2$, we can obtain that for some $\mathbf{c}^* > 0$ and any $u > t_2$

$$\left|\mathbf{q}_2(u,t_2) - \mathbf{q}_2(u,t_1)\right| \le \mathbf{c}^* \int_{t_1}^{t_2} \frac{1}{(u-\theta)^{3/2}} \mathrm{d}\theta \le \mathbf{c}^* \frac{\Delta}{(u-t_2)^{3/2}}\,.$$

Therefore,

$$\left|\mathbf{q}_3(u)\right| \le \mathbf{c}^* \left(\frac{\Delta}{(u-t_2)^{3/2}} + \frac{1}{\sqrt{u-t_1}} \left|\mathbf{q}_1(u,t_2) - \mathbf{q}_1(u,t_1)\right| \right)\,.$$

Note that the definition (3.10) and Proposition 7.1 imply that for any $0 < \varepsilon < 1$

$$\left|\Psi_h(s_2,t) - \Psi_h(s_1,t)\right| \le \mathbf{c}^* |s_2 - s_1|^{\varepsilon}\,, \tag{7.6}$$

where $\mathbf{c}^*$ is some positive constant. So,

$$\left|\mathbf{q}_1(u,t_2) - \mathbf{q}_1(u,t_1)\right| \le \mathbf{c}^*\,(1+|s|^{\varepsilon}) \left(|\mu(u,t_2) - \mu(u,t_1)|^{\varepsilon} + \left|\sigma_1(u,t_2) - \sigma_1(u,t_1)\right|^{\varepsilon} \right).$$

It should be noted here also that for some $\mathbf{c}^* > 0$

$$|\sigma_1(u,t_2) - \sigma_1(u,t_1)| = \frac{|\mathbf{s}(u,t_2) - \mathbf{s}(u,t_1)|}{\sigma_1(u,t_2) + \sigma_1(u,t_1)} \le \mathbf{c}^* \frac{\Delta}{\sqrt{u-t_2}}\,,$$

i.e., for $t_2 < u \le T$

$$\begin{aligned}\left|\mathbf{q}_1(u,t_2)-\mathbf{q}_1(u,t_1)\right| &\le \mathbf{c}^*(1+|s|^{\varepsilon})\Delta^{\varepsilon}\Big(1+\frac{1}{(u-t_2)^{\varepsilon/2}}\Big)\\ &\le \frac{\mathbf{c}^*(1+|s|)\Delta^{\varepsilon}}{(u-t_2)^{\varepsilon/2}}\,.\end{aligned}$$

This implies directly that

$$\begin{aligned}I_2^*(\Delta) &\le \mathbf{c}^*(1+|s|)\int_{t_2+\Delta_1}^{T}\left(\frac{\Delta}{(u-t_2)^{3/2}}+\frac{\Delta^{\varepsilon}}{(u-t_2)^{(\varepsilon+1)/2}}\right)\mathrm{d}u\\ &\le \mathbf{c}^*(1+|s|)\left(\frac{\Delta}{\sqrt{\Delta_1}}+\Delta^{\varepsilon}(\Delta_1)^{(1-\varepsilon)/2}\right)\\ &= \mathbf{c}^*(1+|s|)\left(\Delta^{1-\varepsilon}+\Delta^{2\varepsilon-\varepsilon^2}\right)\end{aligned}$$

and, therefore, for any $0<\varepsilon<1/2$

$$\limsup_{\Delta\to 0}\frac{1}{\Delta^{\varepsilon}}\sup_{s\in\mathbb{R}}\frac{I_1^*(\Delta)+I_2^*(\Delta)}{1+|s|}<\infty\,.$$

Hence, Proposition 7.2. □

8. Verification theorem

Now we give the verification theorem from [2]. Consider on the interval $[0,T]$, the stochastic control process given by the n - dimensional Itô process

$$\mathrm{d}\zeta_t = a(\zeta_t,t,\mathbf{u}_t)\mathrm{d}t + b(t,\zeta_t,\mathbf{u}_t)\mathrm{d}W_t\,, \tag{8.1}$$

where $(W_t)_{0\le t\le T}$ is a standard k - dimensional Brownian motion and the initial value ζ_0 is non random and belongs to is some convex set $\Xi\subseteq\mathbb{R}^n$. We assume that the control process $\mathbf{u}$ takes its values in some convex set $\Theta\subseteq\mathbb{R}^d$. Moreover, we assume that the coefficients a and b satisfy the following conditions:

$\mathbf{A}_1$) *The functions $a(\cdot,t,\cdot)$ and $b(\cdot,t,\cdot)$ are continuous on $\mathbb{R}^n\times\Theta$, $\forall t\in[0,T]$.*
$\mathbf{A}_2$) *For any fixed nonrandom vector $\theta\in\Theta$ and any $x\in\Xi$ the stochastic differential equation (8.1) for $\mathbf{u}_t\equiv\theta$ has a unique strong solution in Ξ.*

Now we introduce admissible control processes for the equation (8.1) on the time interval $[t,T]$ for any $0\le t\le T$. To this end we set the family $\mathbf{F}_t=(\mathcal{F}_{t,s})_{t\le s\le T}$ with $\mathcal{F}_{t,s}=\sigma\{W_v\,,t\le v\le s\}$. A stochastic control process $\mathbf{u}=(\mathbf{u}_s)_{t\le s\le T}$ is called admissible on $[t,T]$ if it is $\mathbf{F}_t$ - progressively measurable with values in Θ, and for any $x\in\Xi$ the equation (8.1) on the time interval $[t,T]$ with $\zeta_t=x$ has a unique strong a.s. continuous solution $(\zeta_v)_{t\le v\le T}$ belonging to Ξ such that

$$\int_t^T(|a(\zeta_v,v,\mathbf{u}_v)|+|b(\zeta_v,v,\mathbf{u}_v)|^2)\mathrm{d}v<\infty\quad\text{a.s.}\,.$$

We denote by $\mathcal{A}_t$ the set of all admissible control processes on the time interval $[t,T]$. Moreover, let $\mathbf{f}$ and $\mathbf{h}$ be continuous utility $\Xi\times[0,T]\times\Theta\to[0,\infty)$ functions. For any $0\le t\le T$ we define the cost function by

$$\boldsymbol{J}(x,t,\mathbf{u}) = \mathbf{E}\left(\int_t^T \mathbf{f}(\zeta_v, v, \mathbf{u}_v)\mathrm{d}v + \mathbf{h}(\zeta_T, T, \mathbf{u}_T)|\zeta_t = x\right).$$

Our goal is to solve the optimization problem

$$\sup_{\mathbf{u}\in\mathcal{V}} \boldsymbol{J}(x,t,\mathbf{u})\,. \tag{8.2}$$

In order to find the solution to (8.2) we investigate the HJB equation

$$\begin{cases} z_t(\zeta,t) + H(\zeta,t,z_\zeta,z_{\zeta\zeta}) = 0, & t\in[0,T]\,,\\ z(\zeta,T) = \mathbf{h}(\zeta), & \zeta\in\mathbb{R}^n\,, \end{cases} \tag{8.3}$$

where

$$H(\zeta,t,z_\zeta,z_{\zeta\zeta}) := \sup_{u\in\Theta}\left(a'(\zeta,t,u)z_\zeta + \frac{1}{2}\mathbf{tr}[bb'(\zeta,t,u)z_{\zeta\zeta}] + \mathbf{f}(\zeta,t,u)\right)\,.$$

Here, $z_t = z_t(\zeta,t)$ denotes the partial derivative of z with respect to t, $z_\zeta = z_\zeta(\zeta,t)$ the gradient vector with respect to ζ in $\mathbb{R}^n$ and $z_{\zeta\zeta} = z_{\zeta\zeta}(\zeta,t)$ denotes the matrix of the second order partial derivatives in the variables ζ. We assume the following conditions hold:

$\mathbf{H}_1$) *There exists a* $\mathbb{R}^n\times[0,T]\to(0,\infty)$ *function* z *from* $\mathbf{C}^{2,1}(\mathbb{R}^n\times[0,T])$, *which satisfies the HJB equation.*

$\mathbf{H}_2$) *There exists a measurable* $\mathbb{R}^n\times[0,T]\to\Theta$ *function* $u_0 = u_0(\zeta,t)$ *such that for all* $\zeta\in\mathbb{R}^n$ *and* $0\le t\le T$,

$$H(\zeta,t,z_\zeta,z_{\zeta\zeta}) = a'(\zeta,t,u_0)z_\zeta + \frac{1}{2}\mathbf{tr}\left(bb'(\zeta,t,u_0)z_{\zeta\zeta}\right) + \mathbf{f}(\zeta,t,u_0)$$

$\mathbf{H}_3$) *There exists a unique strong solution to the Itô equation*

$$d\zeta^*_t = a_0(\zeta^*_t,t)\mathrm{d}t + b_0(\zeta^*_t,t)\mathrm{d}W_t, \quad \zeta^*_0 = x, \quad t\ge 0\,,$$

where $a_0(\zeta,t) = a(\zeta,t,u_0(\zeta,t))$ *and* $b_0(\zeta,t) = b(\zeta,t,u_0(\zeta,t))$.

$\mathbf{H}_4$) *For all* $x\in\mathbb{R}^n$ *and* $0\le t\le T$ *the family* $(z(\zeta^*_\tau,\tau))_{\tau\in\mathcal{M}_t}$ *is uniformly integrable, where* $\mathcal{M}_t$ *is the set of all stopping times with the values in* $[t,T]$.

Theorem 8.1. *Assume that conditions* $\mathbf{H}_1$)- $\mathbf{H}_4$) *hold, then for any* $0\le t\le T$ *the process* $(\mathbf{u}^*_t)_{t\le s\le T}$ *is a solution for the problem* (8.2), *i.e.;*

$$\sup_{\mathbf{u}\in\mathcal{V}} \boldsymbol{J}(x,t,\mathbf{u}) = \boldsymbol{J}(x,t,\mathbf{u}^*)$$

and $z(x,t) = \boldsymbol{J}(x,t,\mathbf{u}^*)$.

9. Cauchy problem

Here we announce the existence theorem from [14] for the Cauchy problem:

$$\begin{cases} u_t - \sum_{1\le i,j\le n} a_{ij}(x,t,u)u_{x_ix_j} + a(x,t,u,u_x) = 0\,,\\ u|_{t=0} = u(x,0) = \psi_0(x)\,, \quad x\in\mathbb{R}^n\,. \end{cases} \tag{9.1}$$

We assume that there exists some functions $(a_1, a_2, ..., a_n)$, such that

$$a_{ij}(x,t,u,p) \equiv \frac{\partial a_i(x,t,u,p)}{\partial p_j}, \quad p \in \mathbb{R}^n. \tag{9.2}$$

Using these functions we set

$$A(x,t,u,p) \equiv a(x,t,u,p) - \sum_{i=1}^{n} \frac{\partial a_i}{\partial u} p_i - \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i}.$$

Moreover, for any $N \geq 1$ we set

$$\Gamma_N = \{(x,t) \in \mathbb{R}^n \times [0,T] : \, |x| \leq N\}.$$

We introduce the following conditions for ensuring the existence of at least one solution $u(x,t)$ for the problem (9.1).

C$_1$) *There exists $\varepsilon > 0$ such that for all $N \geq 1$,*

$$\psi_0(x) \in \mathcal{H}^{2+\varepsilon}(\Gamma_N) \quad \text{and} \quad \max_{x \in \mathbb{R}^n} \mid \psi_0(x) \mid < \infty.$$

C$_2$) There exists $h \geq 0$ and some $\mathbb{R}_+ \to \mathbb{R}_+$ function Φ, such that for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and for all $0 \leq t \leq T$,

$$A(x,t,u,0)u \geq -\Phi(|u|)|u| - b, \quad \text{and} \quad \int_0^\infty \frac{d\tau}{\Phi(\tau)} = +\infty. \tag{9.3}$$

C$_3$) *For $t \in (0,T]$ for arbitrary $x, \; u, \; p \in \mathbb{R}^n$, and any $\xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$, there exists $0 < C_1 \leq C_2$ such that*

$$\sum_{1 \leq i,j \leq n} a_{ij}(x,t,u,p)\xi_i \xi_j \geq 0 \quad \text{and} \quad C_1 |\xi|^2 \leq a_{ij}(x,t,u,p)\xi_i \xi_j \leq C_2 |\xi|^2.$$

C$_4$) *The functions $a_i(x,t,u,p)$ and $a(x,t,u,p)$ are continuous, the functions $(a_i)_{1 \leq i \leq n}$ are differentiable with respect to x, u and $p \in \mathbb{R}^n$, and for any $N \geq 1$*

$$\sup_{(x,t) \leq \Gamma_N} \sup_{|u| \leq N} \sup_{p \in \mathbb{R}^n} \frac{\sum_{i=1}^n \left(|a_i| + |\frac{\partial a_i}{\partial u}|\right)(1+|p|) + \sum_{i,j=1}^n |\frac{\partial a_i}{\partial x_j}| + |a|}{1+|p|^2} < \infty.$$

C$_5$) *For all $N \geq 1$, and for all $(x,t) \in \Gamma_N$, $|u| \leq N$ and $|p| \leq N$, the functions a_i, a, $\partial a_i / \partial p_j$, $\partial a_i / \partial u$, and $\partial a_i / \partial x_i$ are continuous functions satisfying a Hölder condition in x, t, u and p with exponents ε, $\varepsilon/2$, ε and ε respectively for $\varepsilon > 0$ from the condition* **C**$_1$).

Theorem 9.1 *(See Theorem 8.1, p. 495 of [14]). Assume that the conditions* **C**$_1$)*–***C**$_5$) *hold. Then there exists at least one solution $u(x,t)$ of Cauchy problem (9.1) which is bounded in $\mathbb{R}^n \times [0,T]$ and for any $N \geq 1$ belongs to $\mathcal{H}^{2+\varepsilon, 1+\varepsilon/2}(\Gamma_N)$.*

10. Proofs

10.1. Proof of Theorem 5.1

To prove this theorem we use Theorem 8.1, i.e. we need to check the conditions $\mathbf{H}_1) - \mathbf{H}_5)$ of this theorem for the problem (2.6). As to the first condition, note that using Proposition 4.2 one can check directly that the solution of the equation (3.5) can be represented in the form (3.6), where the functions g and Y satisfy the equations (3.7) and (3.9) respectively. Moreover, using the HJB solution (3.6) in (3.4) we calculate the optimal control variables (3.11). Hence $\mathbf{H}_2)$. Then, using these variables in the wealth process (2.2) we obtain the optimal wealth process represented by the stochastic differential equation (3.12) which can be resolved through the Ito formula and represented in the form (3.13). This implies $\mathbf{H}_3)$. Note also, that the function Y in (3.6) is bounded, therefore, the condition $\mathbf{D})$ yields the condition $\mathbf{H}_4)$. So, Theorem 8.1 implies that the strategy (3.14) is optimal and the function (3.6) coincides with the value function (2.6). This means that a solution of the equation (3.9) can be represented through the value function which is unique. Therefore, the equation (3.9) has the unique solution in $\mathbf{C}^{2,1}(\mathbb{R} \times [0,T])$. Hence Theorem 5.1. □

10.2. Proof of Theorem 5.3

We set $\Delta_n(y,t) = h(y,t) - h_n(y,t)$. So, in view of Proposition 6.4 for any $\nu > (\mathbf{r}_*)^2$

$$\sup_{y\in\mathbb{R}\,,\,0\le t\le T} \left(|\Delta_n(y,t)| + |\frac{\partial \Delta_n(y,t)}{\partial y}| \right) \le e^{\nu T} \rho(h,h_n) \le 2(\mathbf{B}_0 + \mathbf{B}_1) \frac{\lambda^n}{1-\lambda}\, e^{\nu T}\,,$$

where $\lambda = \mathbf{r}_*/\sqrt{\nu}$, $\mathbf{B}_0$ and $\mathbf{B}_1$ are defined in (4.6). Therefore, if we take in this upper bound $\nu = n(\mathbf{r}_*)^2$ and $\lambda = 1/\sqrt{n}$, then for the norm defined in (5.3) we obtain that $\|\Delta_n\| = \mathrm{O}(n^{-\delta n})$ as $n \to \infty$ for any $0 < \delta < 1/2$. Hence Theorem 5.3. □

10.3. Proof of Theorem 5.5

First note that, from the definition of the process X^*_n in (5.4) we deduce that

$$\mathrm{d}X^n_t = a^*_n(t) X^n_t \mathrm{d}t + b^*_n(t) X^n_t \mathrm{d}W_t\,, \quad X^n_0 = x\,, \tag{10.1}$$

where $a^*_n(t) = r - \kappa_1 S_t \widetilde{\alpha}_{0,n}(S_t,t) - \widetilde{c}_{0,n}(S_t,t)$ and $b^*_n(t) = \sigma\widetilde{\alpha}_{0,n}(S_t,t)$. Similarly to (3.13), through the Ito formula we obtain that

$$X^n_t = x\,\exp\left\{ \int_0^t b^*_n(u)\mathrm{d}W_u + \int_0^t \left(a^*_n(u) - \frac{1}{2}(b^*_n(u))^2 \right) \mathrm{d}u \right\}. \tag{10.2}$$

Note now that for any $N > 0$ on the set $\{\max_{0\le t\le T} |S_t| \le N\}$ this process coincides with the process $\check{X}_n(t)$ defined in (10.2) by replacing the coefficients $a^*_n(u)$ and $b^*_n(u)$ with $\check{a}_n(u) = a^*_n(u\wedge\sigma_N)$, $\check{b}_n(u) = b^*_n(u\wedge\sigma_N)$ and $\sigma_N = \inf\{t \ge 0 : |S_t| \ge N\} \wedge T$. Note that the functions $\check{a}_n(u)$ and $\check{b}_n(u)$ are bounded, i.e. for some non random $C_N > 0$ the upper bound $\sup_{n\ge1} \max_{0\le t\le T} \left(|\check{a}_n(t)| + |\check{b}_n(t)| \right) \le C_N$. This implies that for some $C_N > 0$

$$\check{X}_n(t) \le C_N\,\exp\left\{ \int_0^t \check{b}_n(u)\mathrm{d}W_u - \frac{1}{2}\int_0^t (\check{b}_n(u))^2 \mathrm{d}u \right\} := C_N \mathcal{E}_t(\check{b}_n)\,,$$

where in this case the stochastic exponential $\mathcal{E}_t(\check{b})$ is a square integrable martingale. Therefore, in view of the Doob inequality we obtain that $\sup_{n\geq 1} \mathbf{E} \sup_{0\leq t\leq T} \mathcal{E}_t^2(\check{b}_n) < \infty$ which implies $\sup_{n\geq 1} \mathbf{E} \max_{0\leq t\leq T} \check{X}_n^2(t) < \infty$. Therefore, taking into account that in this case $\lim_{N\to\infty} \mathbf{P}(\sigma_N < T) = 0$ we get

$$\lim_{N\to\infty} \sup_{n\geq 1} \mathbf{P}(\max_{0\leq t\leq T} X_t^n \geq N) = 0\,. \tag{10.3}$$

In view of Theorem 5.4, to show this theorem it suffices to check that for any $0 < \delta < 1/2$

$$\mathbf{P} - \lim_{n\to\infty} n^{\delta n} \max_{0\leq t\leq T} |\Delta_t^n| = 0\,, \tag{10.4}$$

where $\Delta_t^n = X_t^n - X_t^*$. To this end we set

$$\tau_{n,N} = \inf\{t\geq 0\,:\, |X_t^n| + |X_t^*| \geq N\} \wedge \sigma_N\,.$$

It is clear that the property (10.3) implies

$$\lim_{N\to\infty} \sup_{n\geq 1} \mathbf{P}(\tau_{n,N} < T) = 0\,. \tag{10.5}$$

Therefore, to show (10.4) we have to provide that for any $N > 0$

$$\mathbf{P} - \lim_{n\to\infty} n^{\delta n} \max_{0\leq t\leq \tau_{n,N}} |\Delta_t^n| = 0\,. \tag{10.6}$$

Indeed, from (3.12) and (10.1) it follows that

$$\mathrm{d}\Delta_t^n = Z_{1,n}(t)\mathrm{d}t + Z_{2,n}(t)\mathrm{d}W_t\,, \quad \Delta_0^n = 0\,, \tag{10.7}$$

where $Z_{1,n}(t) = (a_n^*(t)X_t^n - a^*(t)X_t^*)$ and $Z_{2,n}(t) = (b_n^*(t)X_t^n - b^*(t)X_t^*)$. Now we set

$$\delta_n^* = \sup_{s\in\mathbb{R}} \sup_{0\leq t\leq T} \left(\left|\widetilde{\alpha}_0(s,t) - \widetilde{\alpha}_{0,n}(s,t)\right| + \left|\widetilde{c}_0(s,t) - \widetilde{c}_{0,n}(s,t)\right|\right)\,.$$

Then on the set $\{t\leq \tau_{n,N}\}$ we get that $|Z_{1,n}(t)| + |Z_{2,n}(t)| \leq C_N\left(\delta_n^* + |\Delta_t^n|\right)$ for some nonrandom $C_N > 0$. Moreover, the equation (10.7) implies

$$\max_{0\leq u\leq t\wedge\tau_{n,N}} (\Delta_u^n)^2 \leq 2T\int_0^t \widehat{Z}_{1,n}^2(u)\mathrm{d}u + 2\max_{0\leq u\leq t}\left(\int_0^u \widehat{Z}_{2,n}(v)\mathrm{d}W_v\right)^2\,,$$

where $\widehat{Z}_{i,n}(u) = Z_{i,n}(u\wedge\tau_{n,N})$. Setting $\varrho_n(t) = \mathbf{E}\max_{0\leq u\leq t\wedge\tau_{n,N}} \Delta_n^2(u)$ and taking into account that the processes $\widehat{Z}_{1,n}(u)$ and $\widehat{Z}_{2,n}(u)$ are bounded we obtain through the Doob inequality that

$$\varrho_n(t) \leq C_{N,T}\left(\delta_n^* + \int_0^t \varrho_n(u)\mathrm{d}u\right)\,, \quad 0\leq t\leq T\,.$$

Thus by the Gronwall - Bellman lemma (see Lemma A.1 in Appendix A.4) $\varrho_n(T) \leq C_{N,T}\, e^{TC_{N,T}}\,\delta_n^*$. Theorem 5.4 implies the limit (10.6). □

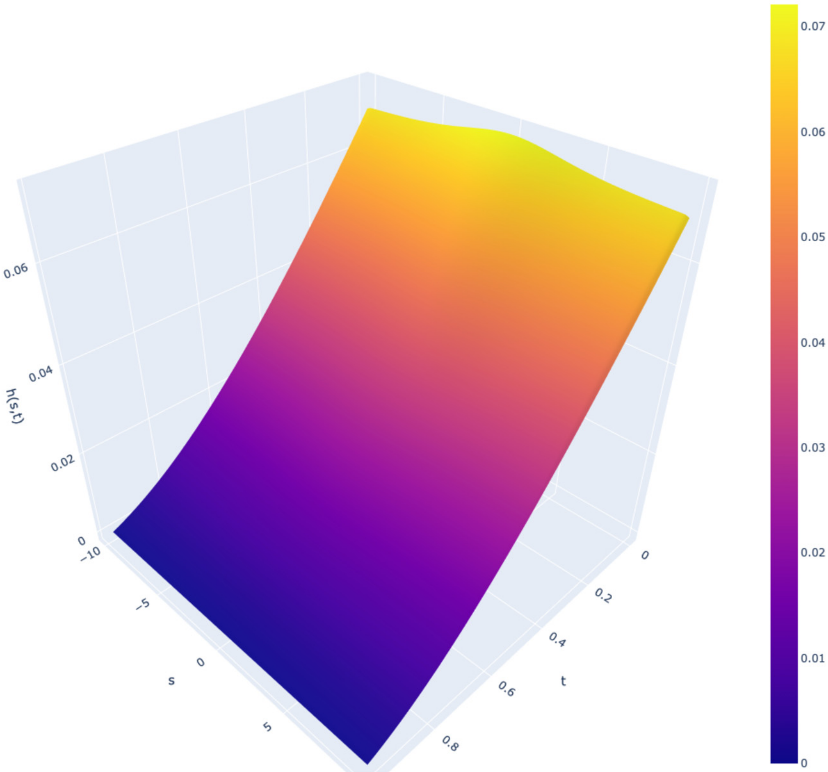

Fig. 1. Fixed point h. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

11. Numerical example

In this section we calculate the strategy (3.14), (3.13) through the Python soft, using the following parameters: $\sigma = 1$, $\gamma = 1/2$, $r = 1/20$, $\kappa = 1/2$, $T = 1$ and $\varpi = 10$. The Feynman–Kac mapping is calculated as

$$\mathcal{L}_h(s,t) = \int_t^T \int_{\mathbb{R}} \Psi_h(z,t)\, \varphi(s,t,z,u)\, \mathrm{d}z\, \mathrm{d}u\,, \tag{11.1}$$

where the function Ψ_h is defined in (3.10) and

$$\varphi(s,t,z,u) = \frac{e^{-\frac{\mathbf{v}^2}{2}}}{\sqrt{2\pi}\sigma_1(u,t)} \quad \text{and} \quad \mathbf{v} = \mathbf{v}(s,t,z,u) = \frac{z - s\mu(u,t)}{\sigma_1(u,t)}\,.$$

The function μ defined in (4.2) can be calculated in the explicit form

$$\mu(u,t) = e^{-\gamma_1 \vartheta_2 (u-t)} \left(\frac{e^{2\gamma_1\vartheta_2 T} - \bar{\theta}_2 e^{2\gamma_1\vartheta_2 u}}{e^{2\gamma_1\vartheta_2 T} - \bar{\theta}_2 e^{2\gamma_1\vartheta_2 t}} \right)^{\rho}$$

where $\rho = \bar{\theta}_1/(2\bar{\theta}_2\vartheta_2\gamma_1)$, $\bar{\theta}_1 = 2\gamma_1\vartheta_2(\vartheta_1 - \vartheta_2)/(\vartheta_1 + \vartheta_2)$ and $\bar{\theta}_2 = (\vartheta_1 - \vartheta_2)(\vartheta_1 + \vartheta_2)$.

In Fig. 1 we calculated the fixed point function h for $n = 4$. In Table 1 we study the convergence for the functions $(h_n)_{n\geq 1}$ calculating the approximation accuracy as

$$\delta_n = \sup_{s,t} |h_n(s,t) - h_{n-1}(s,t)|\,.$$

Table 1
The accuracy δ_n at each iteration.

n	1	2	3	4
δ_n	0,071	0,00026	$6,4 \cdot 10^{-7}$	$1,15 \cdot 10^{-9}$

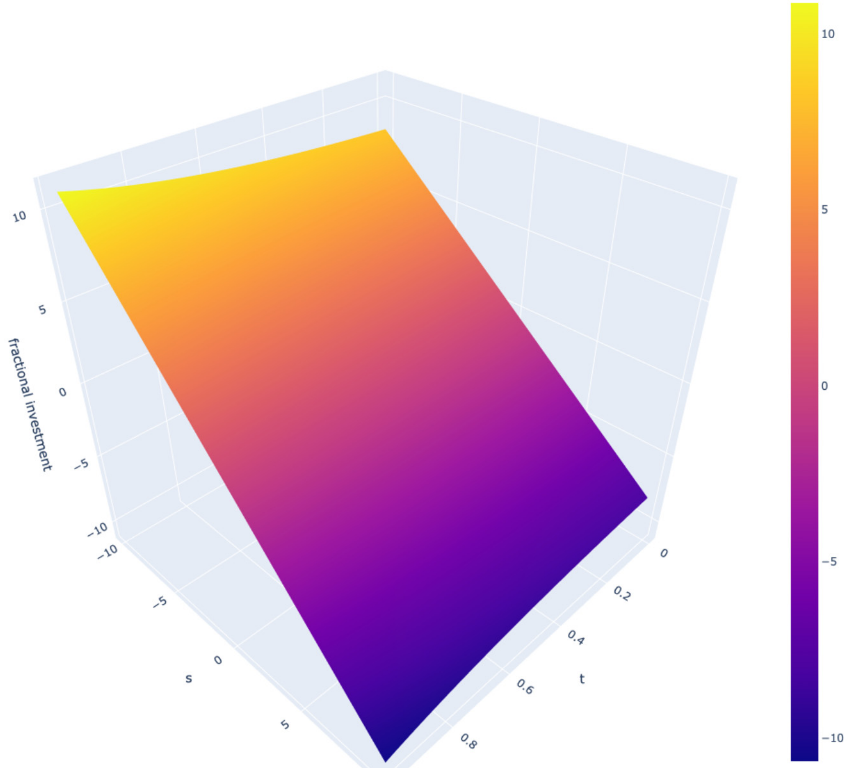

Fig. 2. Fractional investment.

We observe that $\delta_1 \approx 0,07$, $\delta_2 \approx 10^{-3}$, $\delta_3 \approx 10^{-7}$ and $\delta_4 \approx 10^{-9}$. Therefore, one can conclude, that in this example the "super-geometric" effect is well confirmed numerically.

Figs. 2 and 3 represent the fractional strategies (3.11). Figs. 4, 5 and 6 represent the optimal strategy (3.14) with the initial value $x = 10$.

It should be emphasized that the numerical algorithm (5.4) synthesizing the optimal strategies can be implemented very quickly, i.e. we need only four iterations to calculate the fixed point h in Fig. 1 and to construct the strategy. Then note, that the investment strategy explicitly shows in Fig. 4 how much spreads should be bought and sold. As one can see in Figs. 5 and 6 the consumption strategy shows in this example, how much the investor can consume to increase its terminal capital with respect to the initial one, in particular, in this case the terminal capital is $X^*_T \approx 16$ and the initial $X^*_0 = 10$. As to the fractional strategies (3.11) presented in Figs. 2 and 3 we note that in this case the investment increases for the negative spread values and decreases for the positive ones. As to the consumption strategy, we note that the maximal consumption value corresponds to the small spread value for this market. This example numerically illustrates the practical value for the obtained theoretical results.

12. Conclusion

In conclusion, emphasize that in this paper, probably for the first time, the investment and consumption problem for the spread financial markets is studied completely, i.e. it is provided the sufficient condition **D**)

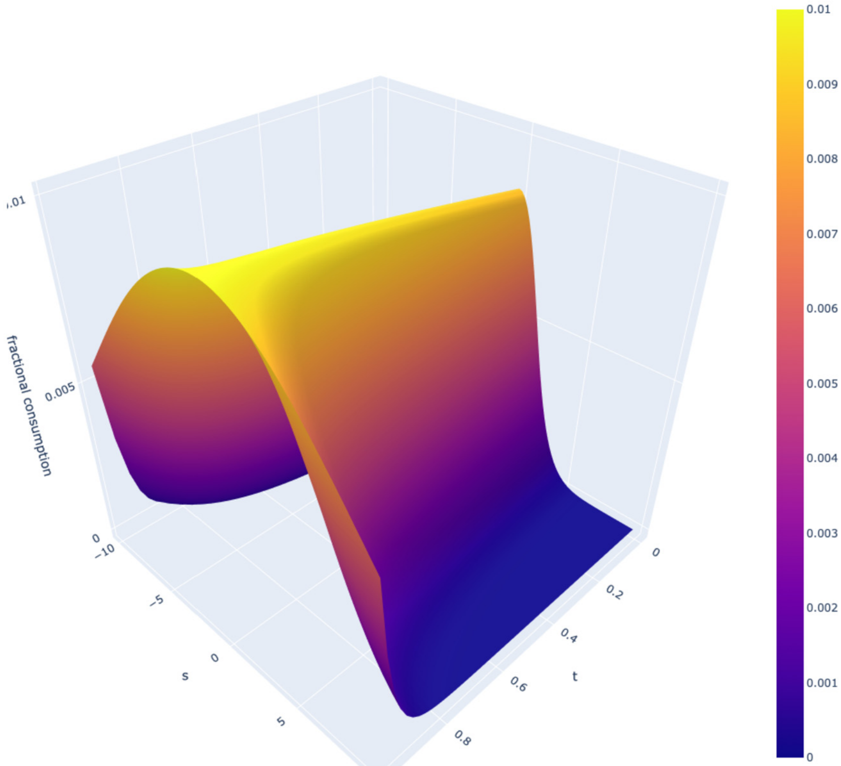

Fig. 3. Fractional consumption.

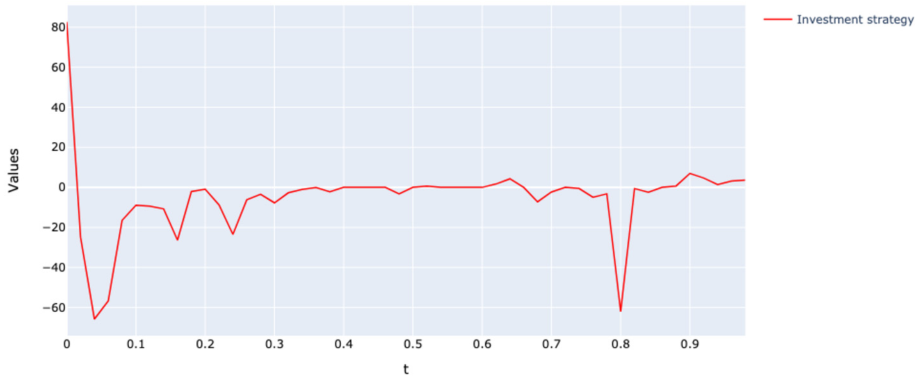

Fig. 4. Sample investment.

under which the optimal strategies are constructed through the verification theorem. It should be noted that this condition is very closed to the necessary, since without the uniform integrability property, generally it is not possible to show that the strategies constructed on the basis of the HJB solution are optimal. In this case, we obtained the new HJB equation for this problem and we found the form for its solution in (3.6). The main difficulty is that the unicity theorem for the parabolic equation (3.9) is shown only among bounded functions, i.e. this equation can have a few unbounded solutions, and we cannot use methods developed for Black-Scholes or stochastic volatility models to analyze the HJB solution. Therefore, it is necessary to develop methods for constructing the bounded solution, which is unique and used in optimal strategies. To this end, we develop the probability methods based on the corresponding Feynman - Kac representation and the fixed point technique in a special metric space and, as a consequence, we establish the convergence rate of a numerical scheme for the optimal strategies which is super geometric. As to the

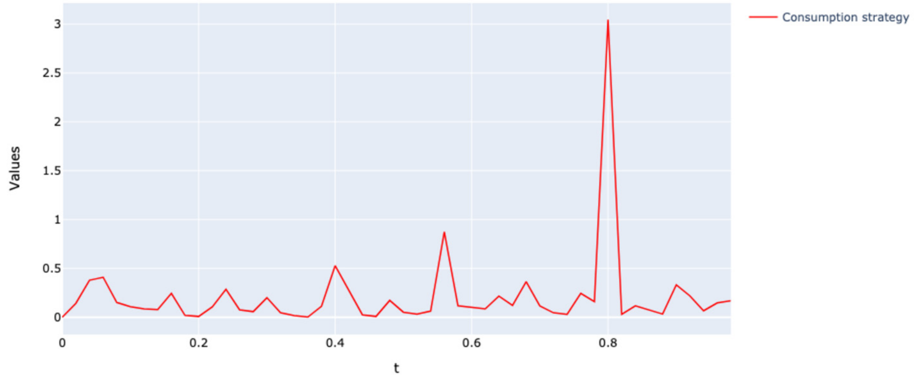

Fig. 5. Sample consumption.

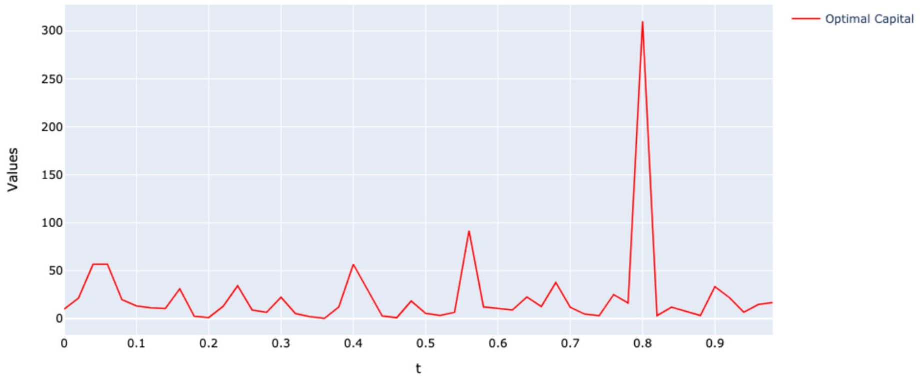

Fig. 6. Sample wealth.

practical point of view the discovered "super geometric rate effect" makes it possible to realize numerical optimal financial control algorithms possessing to increase essentially the speed of information processing and decision making. As a result this will significantly improve the efficiency of using the developed optimal methods for practical investment portfolio calculations under conditions of stochastic financial uncertainty. The economic significance of the obtained results lies in the fact that it is described the financial market conditions under which the optimal investment and consumption strategies are obtained, and the numerical algorithms for their practical implementation are provided. Moreover, in Remark 5.7 we explain how one needs to use the optimal strategies in the spread markets for large time intervals.

Appendix A

A.1. Proof of Proposition 4.2

First, setting $u(s,t) = Y(s, T-t)$, the equation (3.9) can be rewritten as

$$\begin{cases} u_t - \dfrac{\sigma^2}{2} u_{ss} - s\widetilde{g}_1(t)\, u_s - \dfrac{\gamma_1 u_s^2}{2} - a_0(t) - K(s,t)\, e^{-\frac{u}{1-\gamma}} = 0\,, \\ u|_{t=0} = 0\,, \end{cases} \tag{A.1}$$

where the functions $\widetilde{g}_1(t) = g_1(T-t)$, $K(s,t) = \beta_1\, e^{-\frac{s^2 \widetilde{g}(t)}{2(1-\gamma)}}$, $a_0(t) = \sigma^2 \widetilde{g}(t)/2 + r\gamma$ and $\widetilde{g}(t) = g(T-t)$. Note that we can represent the equation (A.1) in the form (9.1) with $n = 1$, $a_1(s,t,u,p) = \sigma^2 p/2$ and

$$A(s,t,u,p) = a(s,t,u,p) = s\widetilde{g}_1(t)\,p - \frac{\gamma_1 p^2}{2} - a_0(t) - K(s,t)\,e^{-\frac{u}{1-\gamma}}\,.$$

Taking into account that $0 \le K(s,t) \le \beta_1$ we obtain that for any $u \in \mathbb{R}$

$$A(s,t,u,0)u \ge -\mathbf{a}_*|u| \quad \text{and} \quad \mathbf{a}_* = \sigma^2 g(0)/2 + r\gamma + \beta_1\,.$$

So, we obtain the condition $\mathbf{C}_2$) with the function $\Phi \equiv \mathbf{a}_*$ and $b = 0$. Taking into account that the conditions $\mathbf{C}_3$)–$\mathbf{C}_5$) can be checked directly we obtain through Theorem 9.1 that the equation (3.9) has a bounded solution. By using this solution and applying the Itô formula to the process (4.1) we can obtain that

$$Y(s,t) = -\int_t^{\tau_n} \Big((Y_t(\eta_u^{s,t},u) + g_1(u)\eta_u^{s,t} Y_s(\eta_u^{s,t},u) + \frac{\sigma^2}{2} Y_{ss}(\eta_u^{s,t},u) \Big)\,\mathrm{d}u$$
$$- \int_t^{\tau_n} Y_s(\eta_u^{s,t},u)\,\mathrm{d}\widetilde{W}_u + Y(\eta_{\tau_n}^{s,t},\tau_n)\,,$$

where $\tau_n = \inf\left\{u \ge t : |\eta_u^{s,t}| \ge n\right\} \wedge T$. Taking into account equation (3.9), we obtain that

$$Y(s,t) = \int_t^{\tau_n} \Psi_Y(\eta_u^{s,t},u)\,\mathrm{d}u - \int_t^{\tau_n} Y_s(\eta_u^{s,t},u)\,\mathrm{d}\widetilde{W}_u + Y(\tau_n,\eta_{\tau_n}^{s,t}).$$

As $\mathbf{E}\int_t^{\tau_n} Y_s(\eta_u^{s,t},u)\,\mathrm{d}\widetilde{W}_u = 0$, we obtain

$$Y(s,t) = \mathbf{E}\int_t^{\tau_n} \Psi_Y(\eta_u^{s,t},u)\,\mathrm{d}u + \mathbf{E}Y(\tau_n,\eta_{\tau_n}^{s,t}).$$

Note here that Y is bounded. So, by Dominated Convergence theorem and in view of the boundary condition in (3.9)

$$\lim_{n\to\infty} \mathbf{E}\,Y(\eta_{\tau_n}^{s,t},\tau_n) = \mathbf{E}\lim_{n\to\infty} Y(\eta_{\tau_n}^{s,t},\tau_n) = \mathbf{E}\,Y(\eta_T^{s,t},T) = 0\,.$$

Reminding here, that $\Psi_Y \ge 0$, the Monotone Convergence theorem yields

$$Y(s,t) = \mathbf{E}\lim_{n\to\infty}\int_t^{\tau_n} \Psi_Y(\eta_u^{s,t},u)\,\mathrm{d}u = \mathbf{E}\int_t^{T} \Psi_Y(\eta_u^{s,t},u)\mathrm{d}u = \mathcal{L}_Y(s,t),$$

i.e. $Y(s,t)$ is a fixed point for $\mathcal{L}$. Moreover, let h be an another FK fixed point function from $\mathcal{X}$. Consider now the following equation

$$f_t(s,t) + \frac{\sigma^2 f_{ss}(s,t)}{2} + s g_1(t) f_s(s,t) + \Psi_h(s,t) = 0, \quad f(s,T) = 0, \tag{A.2}$$

where $g_1(t) = \gamma_1 g(t) - \gamma_2$ and $\Psi_h(s,t)$ is given in (4.3). Similarly to (A.1) denoting $u(s,t) = f(s,T-t)$, we can rewrite the previous equation as

$$u_t(s,t) - \frac{\sigma^2 u_{ss}(s,t)}{2} + a(s,t,u,u_s) = 0, \quad u(s,0) = 0, \tag{A.3}$$

where $a(s,t,u,p) = -sg_1(t)p - \Psi_h(s, T-t)$. Note here, that using the bound (7.1) we get $a(s,t,u,0)u = -\Psi_h(s,T-t)\,u \geq -\Psi^*|u|$ and, therefore, the condition in (9.3) holds with $\Phi(r) \equiv \Psi^*$ and $b = 0$. In view of Propositions 7.1 and 7.2, the function Ψ_h satisfies the Hölder condition $\mathbf{C}_5$) for any $0 < \varepsilon < 1/2$. By using Theorem 9.1 we obtain that equation (A.3) has a bounded solution. Therefore, there exists a bounded solution for the equation (A.2) also, and similarly to the first part of this proof we can obtain that $f = \mathcal{L}_h = h$, i.e. any fixed point for $\mathcal{L}$ from $\mathcal{X}$ is a solution of the equation (3.9). Hence Proposition 4.2. □

A.2. Proof of Proposition 4.3

First of all note, that Propositions 4.3 and 6.4 imply that the equation (3.9) has an unique solution in $\mathcal{X}$ which coincides with the fixed pint h of the mapping (4.3), i.e. the functions $Y(s,t)$ and $Y_s(s,t)$ are bounded. Therefore, taking into account the form (3.6) and the bound (4.4) we note, that to show this proposition it suffices to check that there exists $\mathbf{b} > 1$ such that for any $x > 0$, $s \in \mathbb{R}$ and $0 \leq t \leq T$

$$\sup_{\tau \in \mathcal{M}_t} \mathbf{E}\left((X^*_\tau)^{\mathbf{b}_1}\, e^{\mathbf{b}_2 \frac{S^2_\tau}{\sigma^2}} \,|\, X_t = x\,, S_t = s \right) < \infty\,, \tag{A.4}$$

where $\mathbf{b}_1 = \mathbf{b}\gamma$ and $\mathbf{b}_2 = \mathbf{b}\gamma\kappa_1/2$.

Indeed, using the optimal wealth process (3.12) and the Itô formula we get

$$X^*_t = x \exp\left\{ \int_0^t a^*_1(u)\mathrm{d}u + \int_0^t b^*(u)\mathrm{d}W_u \right\},$$

where $b^*(u) = \sigma\widetilde{\alpha}_0(S_u, u)$ and

$$a^*_1(u) = r - \kappa_1 S_u \widetilde{\alpha}_0(S_u, u) - \widetilde{c}_0(S_u, u) - \frac{\sigma^2 \widetilde{\alpha}^2_0(S_u, u)}{2}\,.$$

Taking into account that

$$\widetilde{\alpha}_0(s,u) = \frac{Y_s(s,u) - s\kappa_1\sigma^{-2}g_2(u)}{1-\gamma} \quad \text{and} \quad g_2(u) = 1 - \sigma^2 g(u)/\kappa_1\,,$$

we get $a^*_1(u) = A(S_u, u)$ and the function A can be represented as

$$A(s,u) = s^2\sigma^{-2}A_0(u) + sA_1(u) + A_2(s,u)\,, \tag{A.5}$$

where

$$A_0(u) = \frac{\kappa_1^2 g_2(u)}{1-\gamma} - \frac{\kappa_1^2 g_2^2(u)}{2(1-\gamma)^2}\,, \quad A_1(u) = \frac{\kappa_1 Y_s(s,u)(g_2(u)-1)}{1-\gamma}$$

and

$$A_2(s,u) = r - \frac{\sigma^2 Y^2_s(s,u)}{2(1-\gamma)^2} - \widetilde{c}_0(s,u)\,.$$

To calculate the conditional expectation (A.4) we have to represent the Ornstein - Uhlenbeck process (2.1) for $u > t$ under the condition $S_t = s$ as

$$S_{t,u} = e^{-\nu(u-t)}s + \sigma\xi_{t,u} \quad \text{and} \quad \xi_{t,u} = \int_t^u e^{-\kappa(u-v)}\mathrm{d}W_v\,.$$

It should be noted here that Proposition 1.1.5 from [10] implies that for any $N > 0$

$$\sup_{\tau\in\mathcal{M}_t} \mathbf{E}\, e^{N|\xi_{t,\tau}|} < \infty\,. \tag{A.6}$$

Therefore, for $\tau \in \mathcal{M}_t$

$$\mathbf{E}\left((X^*_\tau)^{\mathbf{b}_1} \exp\left\{\mathbf{b}_2 \frac{S^2_\tau}{\sigma^2}\right\} \middle| X_t = x, S_t = s\right) = x^{\mathbf{b}_1}\,\mathbf{E}\, e^{\mathbf{U}_{t,\tau}}\,, \tag{A.7}$$

where the exponential power for $v > t$ is defined as

$$\mathbf{U}_{t,v} = \int_t^v \mathbf{b}_1\, a^*_{1,t}(u)\mathrm{d}u + \int_t^v \mathbf{b}_1\, b^*_t(u)\mathrm{d}W_u + \mathbf{b}_2 \frac{S^2_{t,v}}{\sigma^2}$$

with the coefficients $a^*_{1,t}(u) = A^*(S_{t,u}, u)$ and $b^*_t(u) = \sigma\widetilde{\alpha}_0(S_{t,u}, u)$. Now using the representation (A.5) we rewrite these coefficients as

$$a^*_{1,t}(u) = A_0(u)\xi^2_{t,u} + \widetilde{a}_t(u) \quad \text{and} \quad b^*_t(u) = -\frac{\kappa_1}{1-\gamma} g_2(u)\xi_{t,u} + \widetilde{b}_t(u)\,,$$

where $\widetilde{b}_t(u)$ is bounded and $\widetilde{a}_t(u)$ is such that for any $N > 0$,

$$\mathbf{E}\exp\left\{N\int_t^T |\widetilde{a}_t(u)|\,\mathrm{d}u\right\} < \infty\,. \tag{A.8}$$

Therefore, we can write that $S^2_{t,u}/\sigma^2 = \xi^2_{t,u} + \widetilde{S}_{t,u}$, where the process $(\widetilde{S}_{t,u})_{t\le u\le T}$ satisfies the property (A.6). Moreover, taking into account that

$$\mathrm{d}\xi^2_{t,u} = -2\kappa\xi^2_{t,u}\mathrm{d}u + \mathrm{d}u + 2\xi_{t,u}\mathrm{d}W_u \quad \text{and} \quad \xi_{t,t} = 0\,,$$

the exponential power in the left side of the equality (A.7) can be represented as

$$\mathbf{U}_{t,v} = L_{t,v} + \widetilde{S}_{t,v} \tag{A.9}$$

and

$$L_{t,v} = \int_t^v (\mathbf{b}_1 b^*_t(u) + 2\mathbf{b}_2\xi_{t,u})\mathrm{d}W_u + \int_t^v (\mathbf{b}_1 a^*_{1,t}(u) - 2\mathbf{b}_2\kappa\xi^2_{t,u} + \mathbf{b}_2)\mathrm{d}u\,.$$

Note here, that the upper bound (4.4) implies directly that $1-\gamma \le g_2(v) \le 1$. Therefore, we can rewrite the term (A.9) as

$$\mathbf{U}_{t,v} = \int_t^v \overline{g}_1(u)\xi_{t,u}\mathrm{d}W_u + \int_t^v \overline{g}_2(u)\xi^2_{t,u}\mathrm{d}u + \widetilde{\mathbf{U}}_{t,v}\,,$$

where $\overline{g}_1(u) = 2\mathbf{b}_2 - \mathbf{b}_1\kappa_1 g_2(u)/(1-\gamma)$ and

$$\overline{g}_2(u) = \frac{\mathbf{b}_1\kappa_1^2 g_2(u)}{1-\gamma} - \frac{\mathbf{b}_1\kappa_1^2 g_2^2(u)}{2(1-\gamma)^2} - 2\mathbf{b}_2\kappa\,.$$

Moreover, in view of the property (A.6), we can conclude that for any $N > 0$

$$\sup_{\tau\in\mathcal{M}_t} \mathbf{E}\, e^{N\widetilde{\mathbf{U}}_{t,\tau}} < \infty\,.$$

We fix now $p > 1$ which will be chosen later. Then taking into account that

$$\sup_{\tau\in\mathcal{M}_t} \mathbf{E}\, \exp\left\{ p\int_t^\tau \overline{g}_1(u)\xi_{t,u}\mathrm{d}W_u - \frac{p^2}{2}\int_t^\tau \overline{g}_1^2(u)\xi_{t,u}^2\mathrm{d}u \right\} \le 1\,,$$

we obtain through the Hölder inequality with $q = p/(p-1)$, that for any $\tau \in \mathcal{M}_t$

$$\mathbf{E}\exp\left\{\mathbf{U}_{t,\tau}\right\} \le \left(\mathbf{E}\, \exp\left\{ q\int_t^\tau \left(\overline{g}_2(u) + \frac{p}{2}\overline{g}_1^2(u)\right)\xi_{t,u}^2\mathrm{d}u + q\widetilde{\mathbf{U}}_{t,\tau} \right\}\right)^{1/q}\,.$$

One can check directly that

$$\overline{g}_2(u) + \frac{p}{2}\overline{g}_1^2(u) = -t_1\, g_2^2(u) + t_2 g_2(u) + t_3\,,$$

where

$$t_1 = \frac{(1-p\mathbf{b}_1)\mathbf{b}_1\kappa_1^2}{2(1-\gamma)^2}\,,\quad t_2 = \frac{\mathbf{b}_1\kappa_1}{1-\gamma}\left(\kappa_1 - 2p\mathbf{b}_2\right) \quad \text{and} \quad t_3 = 2\mathbf{b}_2\left(p\mathbf{b}_2 - \kappa\right)\,.$$

We chose now $p > 1$ such that $p\gamma < 1$, for example $p = (1+\gamma)/(2\gamma)$. Then, taking $\mathbf{b} = 1/p\gamma$, we obtain $\mathbf{b}_1 = \gamma\mathbf{b} = 1/p$ and $\mathbf{b}_2 = (\mathbf{b}\gamma\kappa_1)/2 = \kappa_1/(2p)$ and

$$\overline{g}_2(u) + \frac{p}{2}\overline{g}_1^2(u) = \frac{\kappa_1(\kappa_1 - 2\kappa)}{2p} \le 0$$

and we come to Proposition 4.3. □

A.3. Proof of Proposition 6.2

Firstly, note that from the definition (4.3) we get that for any $-1 < \delta < 1$,

$$\frac{\mathcal{L}_h(s+\delta,t) - \mathcal{L}_h(s,t)}{\delta} = \int_t^T\int_{\mathbb{R}} \left(\Psi_h(z,u)\left(\frac{\varphi(s+\delta,t,z,u) - \varphi(s,t,z,u)}{\delta}\right)\mathrm{d}z\right)\mathrm{d}u\,,$$

where φ is given in (6.2). Now from (6.2) we obtain that

$$\frac{\varphi(s+\delta,t,z,u) - \varphi(s,t,z,u)}{\delta} = \frac{1}{\delta}\int_s^{s+\delta}\varphi_1(\theta,t,z,u)\mathrm{d}\theta = \varphi_1(s,t,z,u) + \mathbf{D}_\delta(s,t,z,u),$$

where $\mathbf{D}_\delta(s,t,z,u) = \delta^{-1}\int_s^{s+\delta}\Big(\varphi_1(\theta,t,z,u) - \varphi_1(s,t,z,u)\Big)\mathrm{d}\theta$. So, this yields

$$\frac{\mathcal{L}_h(s+\delta,t) - \mathcal{L}_h(s,t)}{\delta} = \int_t^T\int_{\mathbb{R}} \Psi_h(z,u)\varphi_1(s,t,z,u)\mathrm{d}z\,\mathrm{d}u + \mathbf{G}_\delta,$$

where $\mathbf{G}_\delta = \int_t^T\int_{\mathbb{R}}\Psi_h(z,u)\mathbf{D}_\delta(s,t,z,u)\mathrm{d}z\,\mathrm{d}u$. Now we have to prove that $\mathbf{G}_\delta \to 0$ as $\delta \to 0$. As the function $\Psi_h(s,t)$ is bounded for $h \in \mathcal{X}$, therefore,

$$|\mathbf{G}_\delta| \le \Psi^* \int_t^T \frac{1}{|\delta|}\int_{s-|\delta|}^{s+|\delta|} \mathbf{L}(\theta,u)\mathrm{d}\theta\,\mathrm{d}u \le \Psi^*\int_t^T \mathbf{L}^*_\delta(u)\mathrm{d}u\,,$$

where $\Psi^* = \sup_{h\in\mathcal{X}}\sup_{z\in\mathbb{R},\,0\le t\le T}|\Psi_h(z,u)|$, $\mathbf{L}^*_\delta(u) = \max_{s-|\delta|\le\theta\le s+|\delta|}\mathbf{L}(\theta,u)$ and

$$\mathbf{L}(\theta,u) = \int_{\mathbb{R}} |\varphi_1(\theta,t,z,u) - \varphi_1(s,t,z,u)|\mathrm{d}z.$$

From (4.5) and (6.2) we can obtain, that

$$\sup_{-1<\delta<1}\mathbf{L}^*_\delta(u) \le \frac{4\sqrt{2}}{\sigma\sqrt{\pi(u-t)}}\,.$$

Moreover, it is clear, that for any $N > 1$

$$\limsup_{\delta\to 0}\mathbf{L}^*_\delta(u) \le 2\sup_{s-1<\theta<s+1}\int_{|z|>N}|\varphi_1(\theta,t,z,u)|\mathrm{d}z\,.$$

Using the bound (4.5) we get, that for any $s-1<\theta<s+1$ and $u>t$

$$\begin{aligned}\int_{|z|>N}|\varphi_1(\theta,t,z,u)|\mathrm{d}z &\le \frac{1}{\sigma\sqrt{2\pi(u-t)}}\int_{|\sigma_1\mathbf{v}+\theta\mu|>N}|\mathbf{v}|\,e^{-\frac{\mathbf{v}^2}{2}}\mathrm{d}\mathbf{v}\\ &\le \frac{1}{\sigma\sqrt{2\pi(u-t)}}\int_{|\mathbf{v}|>N_1}|\mathbf{v}|\,e^{-\frac{\mathbf{v}^2}{2}}\mathrm{d}\mathbf{v}\,,\end{aligned}$$

where $N_1 = \big(N-(|s|+1)\mu\big)/\sigma_1$. Thus, for any fixed $s\in\mathbb{R}$ and $0\le t<u\le T$ we get that $\lim_{\delta\to 0}\mathbf{L}^*_\delta(u) = 0$. So, by the Lebesgue dominated convergence theorem, $\int_t^T\mathbf{L}^*_\delta(u)\mathrm{d}u \to 0$ as $\delta\to 0$. Hence Proposition 6.2. □

A.4. Grownwall - Bellman lemma

Lemma A.1. *Let ϱ be a non negative bounded $\mathbb{R}\to\mathbb{R}_+$ function such that for some constants $C_1>0$ and $C_2>0$ and for any $0\le t\le T$*

$$\varrho(t) \le C_1 + C_2\int_0^t \varrho(u)\mathrm{d}u\,.$$

Then

$$\max_{0\le t\le T} \varrho(t) \le C_1\, e^{C_2 T}\,. \tag{A.10}$$

Proof. Let now $K>0$ be a some constant such that $\sup_{0\le t\le T}\varrho(t)\le K$. Then by the induction method we can show that for any $n\ge 2$

$$\varrho(t)\le C_1\sum_{j=0}^{n-1}\frac{(C_2 t)^j}{j!}+\frac{K(C_2 t)^n}{n!}\,.$$

Passing here the limit as $n\to\infty$, we get the upper bound (A.10). Hence Lemma A.1. □.

A.5. The smoothness properties of the process (4.1)

Lemma A.2. *For any bounded* $\mathbb{R}\times[0,T]\to\mathbb{R}$ *function* Q *and for* $0<t<T$

$$\left|\frac{\partial}{\partial s}\int_t^T \mathbf{E}Q(\eta_u^{s,t},u)\mathrm{d}u\right|\le\frac{2Q_t^*}{\sigma}\sqrt{\frac{2(T-t)}{\pi}} \quad \textit{and} \quad Q_t^*=\sup_{z\in\mathbb{R}}\sup_{t\le u\le T}|Q(z,u)|\,.$$

Proof. Setting $\bar{Q}(s,u)=\mathbf{E}Q(\eta_u^{s,t},u)$ we note that

$$\bar{Q}(s,u)=\frac{1}{\sqrt{2\pi}\sigma_1(u,t)}\int_{\mathbb{R}} Q(z,u)\exp\left\{-\frac{(z-s\mu(u,t))^2}{2\sigma_1^2(u,t)}\right\}\mathrm{d}z\,.$$

Using here Lebesgue's dominated convergence theorem we can get that

$$\begin{aligned}\frac{\partial}{\partial s}\int_t^T\bar{Q}(s,u)\mathrm{d}u&=\int_t^T\frac{\mu(u,t)}{\sqrt{2\pi}}\int_{\mathbb{R}}Q(z,u)\frac{z-s\mu(u,t)}{\sigma_1^3(u,t)}\,e^{-\frac{(z-s\mu(u,t))^2}{2\sigma_1^2(u,t)}}\,\mathrm{d}z\,\mathrm{d}u\\&=\int_t^T\frac{\mu(u,t)}{\sqrt{2\pi}\sigma_1(u,t)}\int_{\mathbb{R}}Q(s\mu(u,t)+v\sigma_1(u,t),u)v\,e^{-\frac{v^2}{2}}\mathrm{d}v\,\mathrm{d}u\,.\end{aligned}$$

Using here the bound (4.5), we obtain that

$$\left|\frac{\partial}{\partial s}\int_t^T\bar{Q}(s,u)\right|\le\frac{Q_t^*}{\sqrt{2\pi}\sigma}\int_t^T\frac{\mathrm{d}u}{\sqrt{u-t}}\int_{\mathbb{R}}|v|\,e^{-v^2/2}\mathrm{d}v=\frac{2Q_t^*}{\sigma}\sqrt{\frac{2(T-t)}{\pi}}\,.$$

Hence Lemma A.2. □

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