Necessary and sufficient conditions for survival strategies in asset markets with endogenous prices

Alexandra Tokaeva* Mikhail Zhitlukhin†

Abstract

We consider a stochastic asset market model with endogenous asset prices and obtain necessary and sufficient conditions for an agent's strategy to be survival, which means that its share of the total market wealth remains bounded away from zero over an infinite time horizon regardless of the strategies used by other agents. This extends previously known results that focus on construction of particular survival strategies or only necessary conditions for survival.

Keywords: evolutionary mathematical finance, survival strategies, martingales.

1. Introduction

The aim of this paper is to provide a thorough analysis of *survival investment strategies* in an evolutionary model of a financial market with endogenous asset prices. A strategy is called survival if it allows an agent to maintain a share of the total market wealth which is strictly positive and bounded away from zero over an infinite time horizon in any strategy profile, regardless of strategies used by other agents. The interest in studying survival strategies arises from the fact that the presence of agents who use such strategies allows to determine asymptotic characteristics of the market such as asset prices and wealth distribution.

This work continues the strand of the literature emerged in the seminal paper of Blume and Easley (1992) and then further developed by Amir et al. (2005, 2013), Evstigneev et al. (2002), Hens and Schenk-Hoppé (2005) and others. The majority of papers in this direction either construct particular survival strategies in models of various generality or provide necessary conditions for survival. The main contribution of our work consists in that we obtain also sufficient conditions. Furthermore, under mild additional assumptions on the model structure we show that these conditions turn out to be necessary and sufficient.

*Steklov Mathematical Institute of Russian Academy of Sciences, 8 Gubkina st., Moscow, 119991 Russia. Email: aaltokaeva@gmail.com

†Steklov Mathematical Institute of Russian Academy of Sciences, 8 Gubkina st., Moscow, 119991 Russia. Corresponding author. E-mail: mikhailzh@mi-ras.ru

We will work within the model of Amir et al. (2013), which is the most general model of a discrete-time market with short-lived assets which have exogenously specified payoffs and endogenous prices. The market in the model consists of a finite number of agents and a finite number of assets. The assets yield random payoffs distributed among the agents proportionally to the owned shares of the assets. The asset prices in the model are determined endogenously through a short-run equilibrium of supply and demand. Dynamic equilibrium is formed consecutively in each time period in the course of interaction of investment strategies of competing market participants. Assets in the model are called short-lived because it is assumed that at the beginning of each time period they are "reborn" (replaced with their copies). This is a simplified model of a real-world asset market, yet it is frequently used in the literature to study qualitative properties of survival strategies and related questions.

There are two main contributions of the paper. First, we obtain a sufficient condition for survival and show that this condition is necessary if conditional expectations of next-period assets payoffs are strictly positive (this is a very mild modelling assumption). It turns out, that in order to survive, a strategy must be asymptotically close to a certain strategy λ^* that invests in the assets proportionally to the conditional expectations of their relative payoffs. Here we also prove that if at least one agent uses a survival strategy, then the relative asset prices in the model converge to λ^* as time $t \to \infty$. A novelty of the present paper consists in that we introduce the notion of survival on a set of random outcomes, which generalizes the usual notion of survival with probability 1. Strategies surviving on a set with probability strictly between 0 and 1 may arise, for example, in models with structural breaks; see the example provided in Section 3.

Our second contribution pertains to a particular case of the general model in which the asset payoffs are functionally dependent on an ergodic Markov sequence. Such a sequence may represent factors affecting the market. We show that in any strategy profile which contains agents using survival strategies, only those agents survive, while the relative wealth of the other agents asymptotically vanishes. Thus, survival strategies turn out to be "winning" strategies.

Let us briefly mention how this note is related to other works in the literature. A review of papers which investigate survival strategies in models where agents' strategies are modelled directly is provided in Evstigneev et al. (2016). Some more recent results can be found in Amir et al. (2021), Drokin and Zhitlukhin (2020), Evstigneev et al. (2020, 2023), Zhitlukhin (2023a,b). Another large body of literature consists of results on market selection of investment strategies by market forces in the framework of general equilibrium, see, e.g, Alós-Ferrer and Ania (2005), Blume and Easley (2006), Coury and Sciubba (2012), Sandroni (2000), Yan (2008) and references therein. For results which establish links with population genetics, see Lo et al. (2018), Orr (2018). Practical applications for long-term portfolio management can be found in Schnetzer and Hens (2022). It is also worth mentioning that survival strategies can be viewed as *unbeatable* *strategies*. For recent progress on unbeatable strategies and their relation to the models at hand, see Amir et al. (2023).

This paper is organized as follows. In Section 2, we describe the model. In Section 3, we state the definition of a survival strategy and prove the main theorem containing sufficient and necessary conditions for survival. Section 4 is devoted to analysis of the Markov model. The appendix contains known results needed in the proofs.

2. The model

We consider a discrete-time market where $K \geq 2$ assets are traded among $N \geq 2$ agents. The market is influenced by random factors modelled by a probability space with a discrete-time filtration $(\mathcal{F}_t)_{t\geq 0}$.

The assets live for one period and are identically reborn at the beginning of each period. The asset prices are determined endogenously through a short-run equilibrium of supply and demand. The supply (the total volume) of each asset is constant and without loss of generality is normalized to 1. The assets yield random $\mathcal{F}_t$-measurable payoffs $X_{t,k}$ which are distributed among the agents at moments of time $t = 1, 2, \ldots$ We assume that the payoffs are non-negative and satisfy the condition

$$\mathrm{P}(X_{t,k} > 0) > 0, \qquad \sum_{k=1}^{K} X_{t,k} > 0 \text{ a.s.} \tag{1}$$

for all t and k.

Agent $i = 1, \ldots, N$ in this market is characterized by his/her strategy and non-random wealth $w_0^i > 0$ (initial endowment) with which this agent enters the market at time $t = 0$. The wealth w_t^i at further moments of time is random and determined by the dynamics described below.

At every moment of time $t \geq 0$, each agent chooses investment proportions $\lambda_t^i = (\lambda_{t,1}^i, \ldots, \lambda_{t,K}^i)$, according to which he/she allocates the available budget (wealth w_t^i) for purchasing assets at time t, i.e. the budget $\lambda_{t,k}^i w_t^i$ is allocated by agent i for purchasing asset k. The vectors of investment proportions λ_t^i are $\mathcal{F}_t$-measurable, non-negative (short sales are not allowed), and have the sum of their components equal to 1, so λ_t^i assumes values in the set

$$\Delta^K := \{(a_1, \ldots, a_K) \in \mathbb{R}_+^K : a_1 + \ldots + a_K = 1\}.$$

The sequence $\lambda^i = (\lambda_t^i)_{t\geq 0}$ represents the strategy of agent i. Note that we model strategies as functions of a random outcome only, i.e. $\lambda_t^i = \lambda_t^i(\omega)$, $\omega \in \Omega$, yet in the reality agents may employ strategies which take into account actions of other market participants. However, in the context of questions we consider, our modelling assumption does not reduce the generality of the obtained results, see Remark 4 below for details.

By $p_t = (p_{t,1}, \dots, p_{t,K})$, we denote the vector of asset prices at time $t \geq 0$. The prices are formed in equilibrium over each time period as follows. The portfolio of agent i at time t is specified by a vector $x_t^i = (x_{t,1}^i, \dots, x_{t,K}^i)$, where $x_{t,k}^i$ is the amount of asset k in the portfolio. The scalar product $\langle p_t, x_t^i \rangle = \sum_{k=1}^K p_{t,k} x_{t,k}^i$ expresses the value of agent i's portfolio.

At time $t = 0$, the agents' budgets are given by their initial endowments w_0^i. Agent i's budget (wealth) at time $t \geq 1$ is given by

$$w_t^i = \langle x_{t-1}^i, X_t \rangle = \sum_{k=1}^K x_{t-1,k}^i X_{t,k}, \tag{2}$$

i.e. it is constituted of the payoff of the portfolio x_{t-1}^i that was purchased at time $t-1$.

If agent i allocates a fraction $\lambda_{t,k}^i$ of his/her wealth for purchasing asset k at time t, then the number of units of this asset that can be bought is

$$x_{t,k}^i = \frac{\lambda_{t,k}^i w_t^i}{p_{t,k}}. \tag{3}$$

Assume that the market is always in equilibrium: the total asset supply is equal to the total demand (recall that the former is normalized to 1). This implies that for all $t \geq 0$ and $k = 1, \dots, K$ we have

$$1 = \sum_{i=1}^N x_{t,k}^i = \sum_{i=1}^N \frac{\lambda_{t,k}^i w_t^i}{p_{t,k}}.$$

As a result, the equilibrium (market clearing) asset prices are given by

$$p_{t,k} = \sum_{i=1}^N \lambda_{t,k}^i w_t^i. \tag{4}$$

Consequently, as follows from (2)–(4), given a vector of initial endowments $w_0 = (w_0^1, \dots, w_0^N)$ and a strategy profile $\Lambda = (\lambda^1, \dots, \lambda^N)$, the dynamics of agents' wealth in the model is described by the relation

$$w_{t+1}^i = \sum_{k=1}^K \frac{\lambda_{t,k}^i w_t^i}{\sum_{j=1}^N \lambda_{t,k}^j w_t^j} X_{t+1,k}. \tag{5}$$

Formulas (2), (3), (5) make sense only if the asset prices $p_{t,k}$ defined by (4) are non-zero. In view of that, we shall say that a strategy profile is *admissible*, if $p_{t,k} > 0$ for all t and k. One can see that the property of admissibility does not depend on the vector of initial endowments provided that all $w_0^i > 0$.

The following proposition provides a simple sufficient condition of admissibility of a strategy profile.

Proposition 1. *Assume condition* (1) *holds. Then a strategy profile is admissible if* *it contains at least one agent who uses a strategy with all investment proportions being strictly positive with probability 1. In that case, the wealth of this agent always remains strictly positive, and the total market wealth $W_t = \sum_{i=1}^N w_t^i$ satisfies the relation $W_t = \sum_{k=1}^K X_{t,k}$ for all $t \geq 1$.*

The proof is straightforward and we omit it. In what follows, we will always assume that strategy profiles under consideration satisfy the conditions of this proposition.

3. Survival strategies

Let $W_t = \sum_{i=1}^N w_t^i$ denote the total market wealth. We will be interested in the behavior of the *relative wealth* (or the *market shares*) of the agents, which is defined by

$$r_t^i := \frac{w_t^i}{W_t}.$$

The following definition introduces the main concept of the paper. Hereinafter, by $\mathcal{F}$ we denote the σ-algebra of the underlying probability space.

Definition 1. We call a strategy λ *survival* on a set of random outcomes $\Gamma \in \mathcal{F}$, if for any vector of initial endowments w_0 and any strategy profile $\Lambda = (\lambda^1, \dots, \lambda^N)$ consisting of the given strategy $\lambda^i = \lambda$ and arbitrary strategies λ^j of agents $j \neq i$, it holds that

$$\inf_{t \geq 0} r_t^i > 0 \text{ a.s. on } \Gamma.$$

A strategy is called *a.s.-survival* (*survival with probability 1*) if the above relation holds on a set of probability 1.

Remark 1. We say that some relation holds a.s. on a set $\Gamma \in \mathcal{F}$, if the set of random outcomes $\omega \in \Gamma$ where it does not hold has probability 0.

In the model at hand, an a.s.-survival strategy can be constructed explicitly. Let $R_t = (R_{t,1}, \dots, R_{t,K})$ and $\mu_t = (\mu_{t,1}, \dots, \mu_{t,K})$ denote the vector of relative asset payoffs and the vector of their conditional expectations, respectively, i.e.

$$R_{t+1,k} = \frac{X_{t+1,k}}{\sum_{j=1}^K X_{t+1,j}}, \qquad \mu_{t,k} = \mathrm{E}(R_{t+1,k} \mid \mathcal{F}_t).$$

Then from the paper of Amir et al. (2013), it is known that the strategy $\lambda_t^* = \mu_t$ is a.s.-survival. Note that $\mu_{t,k} > 0$ in view of condition (1).

Our first main result characterizes other survival strategies in the model.

Theorem 1. *1) Suppose a strategy λ has strictly positive components ($\lambda_{t,k} > 0$ a.s. for all $k = 1, \dots, K$, $t \geq 0$). Then λ survives on the set*

$$\Gamma = \left\{ \omega : \sum_{t=0}^{\infty} \sum_{k=1}^{K} \mu_{t,k} \ln \frac{\mu_{t,k}}{\lambda_{t,k}} < \infty \right\}. \tag{6}$$

Moreover, if there exists a constant $\varepsilon > 0$ such that $\mu_{t,k} \geq \varepsilon$ for all $t \geq 0$, $k = 1, \ldots, K$, then λ survives on the set[1]

$$\Gamma' = \left\{ \omega : \sum_{t=0}^{\infty} \|\mu_t - \lambda_t\|^2 < \infty \right\}.$$

2) Fix a strategy profile and a vector of initial endowments. Let $\bar{\lambda}_t$ denote the representative strategy *of the agents defined by*

$$\bar{\lambda}_{t,k} = \sum_{i=1}^{N} \lambda_{t,k}^i r_t^i.$$

Suppose that in this strategy profile some agent uses a strategy λ with strictly positive components. Then it holds that

$$\sum_{t=0}^{\infty} \|\mu_t - \bar{\lambda}_t\|^2 < \infty \text{ a.s. on } \Gamma,$$

where Γ is the set defined in (6). In particular, $\lim_{t\to\infty}(\mu_t - \bar{\lambda}_t) = 0$ a.s. on Γ.

3) For any strategy λ surviving on a set Γ, it holds that

$$\sum_{t=0}^{\infty} \|\mu_t - \lambda_t\|^2 < \infty \text{ a.s. on } \Gamma. \tag{7}$$

Moreover, if $\mu_{t,k} \geq \varepsilon > 0$ a.s. on a set Γ for all $t \geq 0$ and $k = 1, \ldots, K$, then condition (7) is necessary and sufficient for a strategy λ with strictly positive components to survive on Γ.

A discussion of the claims of the theorem will be provided after its proof. The proof will be based on some well-known results, which for convenience are relegated to the appendix.

Proof of Theorem 1. 1) Assume agent $i = 1$ uses a strategy λ with strictly positive components. Define the following random sequences U_t and Z_t, $t \geq 0$:

$$U_t = \sum_{s=0}^{t} \sum_{k=1}^{K} \mu_{s,k} \ln \frac{\mu_{s,k}}{\lambda_{s,k}},$$

$$Z_0 = r_0^1, \qquad Z_t = \ln r_t^1 + U_{t-1}, \ t \geq 1.$$

Gibb's inequality implies that $U_t \geq 0$. Observe that the set Γ in formula (6) is nothing but the set of convergence of U_t.

Let us prove that Z_t is a local submartingale. For this end, it will be enough to show that $\mathrm{E}(Z_{t+1}^+ \mid \mathcal{F}_t) < \infty$ and $\mathrm{E}(Z_{t+1} - Z_t \mid \mathcal{F}_t) \geq 0$. The first inequality is clear, since

[1] Here and below, if $\xi = \xi(\omega)$ is a random vector, then by $\|\xi\| = \|\xi(\omega)\|$ we denote its random norm.

$Z_{t+1} \le U_t$. To show the second one, observe that

$$\ln r^1_{t+1} - \ln r^1_t = \ln\left(\sum_{k=1}^K \frac{\lambda_{t,k}}{\bar\lambda_{t,k}} R_{t+1,k}\right) \ge \sum_{k=1}^K R_{t+1,k} \ln \frac{\lambda_{t,k}}{\bar\lambda_{t,k}},$$

where we used the convexity of the logarithm and treated $R_{t+1,k}$ as coefficients of the convex combination of the numbers $\lambda_{t,k}/\bar\lambda_{t,k}$. This implies the bound

$$\mathrm{E}(\ln r^1_{t+1} - \ln r^1_t \mid \mathcal{F}_t) \ge \sum_{k=1}^K \mu_{t,k} \ln \frac{\lambda_{t,k}}{\bar\lambda_{t,k}}.$$

From the definition of Z_t, we find

$$\mathrm{E}(Z_{t+1} - Z_t \mid \mathcal{F}_t) \ge \sum_{k=1}^K \mu_{t,k} \ln \frac{\mu_{t,k}}{\bar\lambda_{t,k}} \ge 0. \tag{8}$$

Consequently, Z is a local submartingale.

Since $Z_t \le U_{t-1}$ and U_t converges on the set Γ, the limit $\lim_{t\to\infty} Z_t$ exists on Γ (see the appendix for details on convergence of martingales). As a result, $\rho = \lim_{t\to\infty} \ln r^1_t$ also exists on Γ, which implies that $\lim_{t\to\infty} r^1_t = e^\rho > 0$, hence λ survives on Γ.

To prove that λ survives on Γ' provided that the condition $\mu_{t,k} \ge \varepsilon > 0$ holds true, we can use the following bounds which follow from reverse Pinsker's inequality (the notation $|\cdot|$ below stands for the ℓ^1-norm of a vector):

$$\sum_{k=1}^K \mu_{t,k} \ln \frac{\mu_{t,k}}{\lambda_{t,k}} \le \frac{|\mu_t - \lambda_t|^2}{2\min_k \lambda_{t,k}} = O(\|\mu_t - \lambda_t\|^2) \text{ as } t \to \infty \text{ on } \Gamma',$$

where we used that $\|\mu_t - \lambda_t\| \to 0$ on Γ', so $\min_k \lambda_{t,k}$ is asymptotically bounded away from zero. Then $\Gamma' \subseteq \Gamma$, which implies the survival property on Γ'.

2) Using Pinsker's inequality, we can improve the second inequality in equation (8) as follows:

$$\mathrm{E}(Z_{t+1} - Z_t \mid \mathcal{F}_t) \ge \frac{1}{2}|\mu_t - \bar\lambda_t|^2.$$

Consequently, the compensator A_t of the submartingale Z_t is bounded from below:

$$A_t \ge \frac{1}{2}\sum_{s=0}^{t-1} |\mu_s - \bar\lambda_s|^2.$$

Because Z_t is bounded from above by U_{t-1} and the latter sequence converges on Γ, the compensator A_t also converges on Γ, which yields the second claim of the theorem.

3) If a strategy λ survives on a set Γ, then it must also survive on this set if placed in the profile $\Lambda = (\lambda^*, \lambda)$, where $\lambda^*_t = \mu_t$. For this strategy profile, we have $\mu_t - \bar\lambda_t =$

$(1-r_t^1)(\mu_t - \lambda_t)$. Then, according to the second claim of the theorem, it holds that

$$\sum_{t=0}^{\infty}(1-r_t^1)^2\|\mu_t-\lambda_t\|^2 < \infty \text{ a.s.},$$

because λ^* is an a.s.-survival strategy. But since λ survives as well, the relative wealth r_t^1 is asymptotically bounded away from 1 on the set Γ, which implies the convergence of the series $\sum_{t=0}^{\infty}\|\mu_t-\lambda_t\|^2$ on Γ.

If $\mu_{t,k} \geq \varepsilon > 0$ a.s. on Γ, then the sufficiency of condition (7) for survival on Γ follows from the first claim of the theorem. □

Remark 2. In view of Proposition 1, one can see that the representative strategy $\bar{\lambda}_t$ coincides with the relative asset prices

$$\bar{\lambda}_{t,k} = \bar{p}_{t,k} := \frac{p_{t,k}}{\sum_{j=1}^{K} p_{t,j}}.$$

As a result, the second claim of Theorem 1 shows that the relative asset prices converge to μ_t on Γ.

Remark 3. In the literature, the notion of survival is typically formulated as survival with probability 1. Let us provide a natural example of a strategy surviving on a set with probability strictly between 0 and 1.

Suppose the asset payoffs are given by the sequence of random vectors

$$X_t = X_t^{(1)}\mathrm{I}(t < \theta) + X_t^{(2)}\mathrm{I}(t \geq \theta),$$

where $X_t^{(1)}$ and $X_t^{(2)}$ are sequences of i.i.d. random vectors and θ is a random moment of time with values in $\mathbb{Z}_+ \cup \{\infty\}$ representing a *structural break* (a *changepoint*) in the sequence of payoffs. The random event $\{\omega : \theta = \infty\}$ corresponds to the absence of a structural break and may have positive probability. If the mean vectors $\mu^{(1)} = \mathrm{E}X_t^{(1)}$ and $\mu^{(2)} = \mathrm{E}X_t^{(2)}$ are different, then the strategy $\lambda_t = \mu^{(1)}$ survives on $\Gamma = \{\omega : \theta = \infty\}$, while it does not survive on the complement of Γ if at least one agent in the strategy profile uses the strategy $\lambda'_t = \mu^{(2)}$ (the latter claim follows from Theorem 2 in the next section).

Remark 4. Let us explain how Theorem 1 can be extended to allow more general strategies. Consider strategies that depend on past actions of agents and initial endowments and can be written as $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^N) \otimes \mathcal{B}((\Delta^K)^N) \otimes \ldots \otimes \mathcal{B}((\Delta^K)^N)$-measurable functions $\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_t(\omega, w_0, h_0, \ldots, h_{t-1})$. The argument $w_0 = (w_0^1, \ldots, w_0^N)$ stands for the vector of initial endowments, and $h_s = (\lambda^{ik})_{i,k}$, where $i = 1, \ldots, N$, $k = 1, \ldots, K$, for the investment proportions chosen by the agents at time s. These functions assume values in Δ^K, so that $\boldsymbol{\lambda}_t^i$ is the proportion of wealth invested in asset i.

Given a vector of initial endowments and a strategy profile $\Lambda = (\boldsymbol{\lambda}^1, \dots, \boldsymbol{\lambda}^N)$, we can generate recursively the sequence of $\mathcal{F}_t$-measurable functions

$$\lambda_t^i(\omega) = \boldsymbol{\lambda}_t^i(\omega, w_0, h_0(\omega), \dots, h_{t-1}(\omega)), \tag{9}$$

where $h_s(\omega) = (\lambda_s^{ik}(\omega))_{i,k}$. The functions λ_t^i represent the investment proportions that are chosen by the agents in this particular strategy profile. Then we define the wealth dynamics by the same formula (5) as above and say that a strategy is survival, if $\inf_{t\geq 0} r_t^i > 0$ a.s. in any strategy profile in which agent i uses this strategy.

Let us call a strategy *history-independent*, if each $\boldsymbol{\lambda}_t$ is a function of ω only. Then it is easy to see that if a history-independent strategy is survival in the the class of all history-independent strategies (as in Definition 1), then it is also survival in the class of all strategies.

As a result, Theorem 1 gives necessary and sufficient conditions for a history-independent strategy to be survival. Moreover, it implies that if a strategy profile contains a history-independent survival strategy, then the representative strategy $\bar{\lambda}_t$ converges to μ_t regardless of whether other strategies depend on the history or not.

4. Results for the Markov model

In this section, under the following additional assumption, we will show that λ^* is not only a survival, but a "winning" strategy.

Assumption 1. Let the sequence of relative payoffs R_t satisfy the following conditions.

(a) There exists an ergodic stationary Markov sequence $s = (s_t)_{t\geq 0}$ with values in some measurable space $\mathcal{S}$ such that R_t functionally depend on s_t, i.e. $R_t = R(s_t)$, where $R\colon \mathcal{S} \to \Delta^K$ is a non-random measurable function;

(b) $\mathrm{E}(\ln \mu_k(s_t))^2 < \infty$ for each $k = 1, \dots, K$, where $\mu_k(s) = \mathrm{E}(R_{t+1,k} \mid s_t = s)$;

(c) $R_{t+1,k}$, $k = 1, \dots, K$, are not conditionally linearly dependent, i.e. no linear combination of them with $\sigma(s_t)$-measurable coefficients is a null random variable, except when all the coefficients are a.s. null.

The sequence s_t can be interpreted as a sequence of factors which affect the market in the model, or a sequence of "states of the world."

According to the previous section, the strategy $\lambda_t^* = \mu(s_t)$ is a.s.-survival. The goal of the next theorem is to prove a stronger result. Suppose a strategy profile consists of strategies of the form $\lambda_t = \lambda(s_t)$, where λ are non-random measurable functions. We will show that if at least one agent uses the strategy λ^*, then the relative wealth of any agent with a different strategy vanishes asymptotically and, hence, the total relative wealth of agents who use λ^* converges to 1 as $t \to \infty$.

Theorem 2. *Let Assumption 1 be satisfied. Consider a strategy profile* $\Lambda = (\lambda^1, \dots, \lambda^N)$, *in which every agent uses a strategy of the form* $\lambda_t^i = \lambda^i(s_t)$, *where* λ^i *are non-random functions. Assume that* $\lambda_k^i(s_t) \geq \varepsilon(s_t) > 0$ *a.s. for all* i, k *with a function* $\varepsilon(s)$ *satisfying the condition* $\mathrm{E}(\ln \varepsilon(s_t))^2 < \infty$.

If at least one agent uses the strategy $\lambda_t^* = \mu(s_t)$ *and* $\mathrm{P}(\lambda^i(s_t) \neq \mu(s_t)) > 0$ *for some agent* i, *then* $\lim_{t\to\infty} r_t^i = 0$ *a.s.*

Remark 5. For the particular case of the model at hand where s_t is a sequence of i.i.d. variables, the domination property of the strategy λ^* (i.e. that its relative wealth tends to 1) was proved by Evstigneev et al. (2002). In that case, λ^* is a constant strategy.

The domination property in a Markov model was proved by Amir et al. (2005) under an assumption that the representative strategy of the other agents (called the *CAPM rule* in that paper) differs from λ^*. Compared to that, our theorem relies on the weaker assumption, which concerns only individual strategies.

Proof. Without loss of generality, we may assume that the underlying filtration $(\mathcal{F}_t)_{t\geq 0}$ is generated by the sequence s_t.

Let agent 1 use the strategy λ^*. Then $\lim_{t\to\infty} r_t^1 > 0$ a.s., and it will be enough to show that $\lim_{t\to\infty} r_t^1/r_t^i = \infty$ a.s. for any agent i who uses a strategy different from λ^*. For that end, we will prove the inequality

$$\liminf_{t\to\infty} \frac{1}{t} \ln \frac{r_t^1}{r_t^i} > 0 \text{ a.s.}$$

Denote $D_t = \ln(r_t^1/r_t^i) - \ln(r_{t-1}^1/r_{t-1}^i)$. Then we have the obvious representation

$$\frac{1}{t} \ln \frac{r_t^1}{r_t^i} = \frac{1}{t} \ln \frac{r_0^1}{r_0^i} + \frac{1}{t} \sum_{u=0}^{t-1} (D_{u+1} - \mathrm{E}(D_{u+1} \mid \mathcal{F}_u)) + \frac{1}{t} \sum_{u=0}^{t-1} \mathrm{E}(D_{u+1} \mid \mathcal{F}_u). \tag{10}$$

We will show that the limit of the second term in the right-hand side is zero, and the limit inferior of the third term is strictly positive.

Let us begin with the second term. We have

$$D_{t+1} = \ln \left(\sum_{k=1}^K \frac{\mu_k(s_t)}{\bar{\lambda}_{t,k}} R_{t+1,k} \right) - \ln \left(\sum_{k=1}^K \frac{\lambda_k^i(s_t)}{\bar{\lambda}_{t,k}} R_{t+1,k} \right). \tag{11}$$

Observe that $D_{t+1} \leq -2 \ln \varepsilon(s_t)$, which follows from the bounds $\mu_k(s_t) \leq 1$ and $\bar{\lambda}_{t,k} \geq \varepsilon(s_t)$ applied to the first logarithm, and the bounds $\lambda_k^i(s_t) \geq \varepsilon(s_t)$ and $\bar{\lambda}_{t,k} \leq 1$ applied to the second logarithm. In the same way, we obtain the inequality $D_{t+1} \geq 2 \ln \varepsilon(s_t)$.

Consequently, since we assume that $(\ln \varepsilon(s_t))^2$ is integrable, the sequence $\xi_t = \sum_{u=0}^{t-1} (D_{u+1} - \mathrm{E}(D_{u+1} \mid \mathcal{F}_u))$ is a square integrable martingale. For its quadratic characteristic, we have

$$\langle \xi \rangle_t = \sum_{u=0}^{t-1} \mathrm{E}((D_{u+1} - \mathrm{E}(D_{u+1} \mid \mathcal{F}_u))^2 \mid \mathcal{F}_u) \le \sum_{u=0}^{t-1} \mathrm{E}(D_{u+1}^2 \mid \mathcal{F}_u) \le 4 \sum_{u=0}^{t-1} (\ln \varepsilon(s_u))^2.$$

From this estimate and Birkhoff's ergodic theorem, it follows that $\limsup_{t\to\infty} \langle \xi \rangle_t / t < \infty$. Then Doob's convergence theorem and the strong law of large numbers for square integrable martingales imply that $\lim_{t\to\infty} \xi_t / t = 0$ a.s.

Now consider the third term in the right-hand side of (10). In order to prove that its limit inferior is strictly positive, we will show that there exists a random sequence V_t and a function $g(s)$ such that $\lim_{t\to\infty} V_t = 0$ a.s., $\mathrm{E}g(s_t) > 0$, and for all $t \ge 1$

$$\mathrm{E}(D_{t+1} \mid \mathcal{F}_t) \ge V_t + g(s_t). \tag{12}$$

If these properties hold true, then the rest of the proof will follow from the convergence of the Cesàro mean (applied to V_t) and the ergodic theorem (applied to $g(s_t)$).

Using the concavity of the logarithm and Pinsker's inequality in (11), one can see that the conditional expectation of the first logarithm is non-negative, and therefore

$$\begin{aligned} \mathrm{E}(D_{t+1} \mid \mathcal{F}_t) &\ge -\mathrm{E}\left(\ln \left(\sum_{k=1}^{K} \frac{\lambda_k^i(s_t)}{\bar\lambda_{t,k}} R_{t+1,k} \right) \,\middle|\, \mathcal{F}_t \right) \\ &\ge -\ln \left(\max_{k=1,\dots,K} \frac{\mu_k(s_t)}{\bar\lambda_{t,k}} \right) - \mathrm{E}\left(\ln \left(\sum_{k=1}^{K} \frac{\lambda_k^i(s_t)}{\mu_k(s_t)} R_{t+1,k} \right) \,\middle|\, \mathcal{F}_t \right). \end{aligned}$$

This implies that inequality (12) holds for the sequence

$$V_t = -\ln \left(\max_{k=1,\dots,K} \frac{\mu_k(s_t)}{\bar\lambda_{t,k}} \right)$$

and the function

$$g(s) = -\mathrm{E}\left(\ln \left(\sum_{k=1}^{K} \frac{\lambda_k^i(s)}{\mu_k(s)} R_{t+1,k} \right) \,\middle|\, s_t = s \right).$$

By Theorem 1, we have $\lim_{t\to\infty} \|\mu(s_t) - \bar\lambda_t\| = 0$ a.s. and hence $\lim_{t\to\infty} V_t = 0$. Let us show that $\mathrm{E}g(s_t) > 0$.

Observe that Assumption 1(c) implies that for any $t \ge 0$ and any $\sigma(s_t)$-measurable random variables $c_1, \dots, c_K$ such that $\mathrm{P}(c_i \ne c_j) > 0$ for at least one pair (i, j), the random variable $\zeta = \sum_{k=1}^{K} c_k R_{t+1,k}$ is not constant. Indeed, if $c_0 := \sum_{k=1}^{K} c_k R_{t+1,k}$ is constant, then $\sum_{k=1}^{K} (c_k - c_0) R_{t,k} = 0$ and, hence, all c_k must be equal to c_0 by Assumption 1(c), which is a contradiction.

Consequently, using Jensen's inequality, we find

$$\mathrm{E} g(s_t) = -\mathrm{E} \ln \left(\sum_{k=1}^{K} \frac{\lambda_k^i(s_t)}{\mu_k(s_t)} R_{t+1,k} \right) > -\ln \left(\mathrm{E} \sum_{k=1}^{K} \frac{\lambda_k^i(s_t)}{\mu_k(s_t)} R_{t+1,k} \right) = 0,$$

where the strict inequality takes place in view of the strict concavity of the logarithm and that its argument is not constant. Thus, $\mathrm{E} g(s_t) > 0$, which finishes the proof. □

5. Appendix

In this appendix, we briefly state some known results which were needed in the proofs. Let $|x|$ denote the ℓ^1-norm of a vector $x \in \mathbb{R}^K$, i.e. $|x| = \sum_{k=1}^K |x^k|$. Then for any $x, y \in \Delta^K$ such that $x^k, y^k > 0$ for all $k = 1, \dots, K$, it holds that

$$\frac{1}{2}|x-y|^2 \le \sum_{k=1}^{K} x^k \ln \frac{x^k}{y^k} \le \frac{|x-y|^2}{2 \min\limits_{k=1,\dots,K} y^k}.$$

The first inequality is the well-known *Pinsker's inequality* for the Kullback–Leibler distance (the middle part of the formula) and the total variation distance ($\frac{1}{2}|x-y|$), if one considers x and y as discrete probability distributions on a set of K elements (see, e.g., Cover and Thomas (2006), Lemma 11.6.1). The second inequality is known as *reverse Pinsker's inequality* (see, e.g., Sason and Verdú (2015)).

Gibbs' inequality states that the Kullback–Leibler distance is non-negative, i.e. for vectors x, y as in the above lemma it holds that

$$\sum_{k=1}^{K} x^k \ln \frac{x^k}{y^k} \ge 0.$$

Moreover, the equality takes place if and only if $x = y$. Note that Gibbs' inequality is a simple corollary from Pinsker's inequality.

Next we provide results from the theory of martingales. More details can be found in Chapter 7 of Shiryaev (2019) or in Liptser and Shiryaev (1989).

Assume given a probability space with a discrete-time filtration $(\mathcal{F}_t)_{t\ge 0}$. Recall that a *local martingale* (or a *local submartingale*) is an adapted sequence $(X_t)_{t\ge0}$ such that $\mathrm{E}|X_0| < \infty$ and there exists a non-decreasing sequence of stopping times $(\tau_k)_{k\ge0}$ such that $\lim_{k\to\infty} \tau_k = \infty$ a.s. and the stopped sequence $X_t^{\tau_k} = X_{\tau_k \wedge t}$ is a martingale for each k (respectively, a submartingale). Equivalently, an adapted sequence X with $\mathrm{E}|X_0| < \infty$ is a local martingale (local submartingale) if $\mathrm{E}(X_t^+ \mid \mathcal{F}_{t-1}) < \infty$ and $\mathrm{E}(X_t \mid \mathcal{F}_{t-1}) = X_{t-1}$ for all $t \ge 1$ (respectively, $\mathrm{E}(X_t \mid \mathcal{F}_{t-1}) \ge X_{t-1}$).

Doob's decomposition implies that any local submartingale can be uniquely decomposed as $X_t = X_0 + M_t + A_t$, where M is a local martingale, A is a non-decreasing predictable sequence (the compensator of X), $M_0 = A_0 = 0$.

As a corollary from *Doob's convergence theorem*, it follows that if X is a local sub-martingale with compensator A such that $X_t \leq U_{t-1}$ for all $t \geq 0$, where U is an adapted random sequence, then on the set $\{\omega : \sup_{t\geq 0} U_t < \infty\}$ a.s. there exist finite limits $X_\infty := \lim_{t\to\infty} X_t$ and $A_\infty := \lim_{t\to\infty} A_t$.

The *quadratic characteristic* of a square integrable martingale X is the unique non-decreasing predictable sequence $\langle X\rangle_t$ such that $\langle X\rangle_0 = 0$ and $X_t^2 - \langle X\rangle_t$ is a martingale. In the explicit form,

$$\langle X\rangle_t = \sum_{s=1}^{t} \mathrm{E}((X_s - X_{s-1})^2 \mid \mathcal{F}_{s-1}).$$

If X is a square integrable martingale, then the limit $X_\infty = \lim_{t\to\infty} X_t$ exists on the set $\{\omega : \lim_{t\to\infty}\langle X\rangle_t < \infty\}$.

The *strong law of large numbers* for square integrable martingales states that

$$\lim_{t\to\infty} \frac{X_t}{\langle X\rangle_t} = 0 \text{ a.s. on the set } \{\omega : \lim_{t\to\infty} \langle X\rangle_t = \infty\}.$$

References

Alós-Ferrer, C. and Ania, A. B. (2005). The asset market game. *Journal of Mathematical Economics*, 41(1-2):67–90.

Amir, R., Evstigneev, I. V., Hens, T., Potapova, V., and Schenk-Hoppé, K. R. (2021). Evolution in pecunia. *Proceedings of the National Academy of Sciences*, 118(26):e2016514118.

Amir, R., Evstigneev, I. V., Hens, T., and Schenk-Hoppé, K. R. (2005). Market selection and survival of investment strategies. *Journal of Mathematical Economics*, 41(1-2):105–122.

Amir, R., Evstigneev, I. V., and Potapova, V. (2023). Unbeatable strategies. *Economic theory*, pages 1–30.

Amir, R., Evstigneev, I. V., and Schenk-Hoppé, K. R. (2013). Asset market games of survival: a synthesis of evolutionary and dynamic games. *Annals of Finance*, 9(2):121–144.

Blume, L. and Easley, D. (1992). Evolution and market behavior. *Journal of Economic Theory*, 58(1):9–40.

Blume, L. and Easley, D. (2006). If you're so smart, why aren't you rich? Belief selection in complete and incomplete markets. *Econometrica*, 74(4):929–966.

Coury, T. and Sciubba, E. (2012). Belief heterogeneity and survival in incomplete markets. *Economic Theory*, 49(1):37–58.

Cover, T. M. and Thomas, J. A. (2006). *Elements of Information Theory*. John Wiley & Sons, 2nd edition edition.

Drokin, Y. and Zhitlukhin, M. (2020). Relative growth optimal strategies in an asset market game. *Annals of Finance*, 16:529–546.

Evstigneev, I., Hens, T., Potapova, V., and Schenk-Hoppé, K. R. (2020). Behavioral equilibrium and evolutionary dynamics in asset markets. *Journal of Mathematical Economics*, 91:121–135.

Evstigneev, I., Hens, T., and Schenk-Hoppé, K. R. (2016). Evolutionary behavioral finance. In Haven, E. et al., editors, *The Handbook of Post Crisis Financial Modelling*, pages 214–234. Palgrave Macmillan UK.

Evstigneev, I., Tokaeva, A., Vanaei, M., and Zhitlukhin, M. (2023). Survival strategies in an evolutionary finance model with endogenous asset payoffs. *Annals of Operations Research*, pages 1–21.

Evstigneev, I. V., Hens, T., and Schenk-Hoppé, K. R. (2002). Market selection of financial trading strategies: Global stability. *Mathematical Finance*, 12(4):329–339.

Hens, T. and Schenk-Hoppé, K. R. (2005). Evolutionary stability of portfolio rules in incomplete markets. *Journal of Mathematical Economics*, 41(1-2):43–66.

Liptser, R. S. and Shiryaev, A. N. (1989). *Theory of Martingales.* Kluwer Academic Publishers.

Lo, A. W., Orr, H. A., and Zhang, R. (2018). The growth of relative wealth and the kelly criterion. *Journal of Bioeconomics*, 20:49–67.

Orr, H. A. (2018). Evolution, finance, and the population genetics of relative wealth. *Journal of Bioeconomics*, 20:29–48.

Sandroni, A. (2000). Do markets favor agents able to make accurate predictions? *Econometrica*, 68(6):1303–1341.

Sason, I. and Verdú, S. (2015). Upper bounds on the relative entropy and Rényi divergence as a function of total variation distance for finite alphabets. In *2015 IEEE Information Theory Workshop-Fall (ITW)*, pages 214–218. IEEE.

Schnetzer, M. and Hens, T. (2022). Evolutionary finance for multi-asset investors. *Financial Analysts Journal*, 78(3):115–127.

Shiryaev, A. N. (2019). *Probability–2.* Springer, 3rd edition.

Yan, H. (2008). Natural selection in financial markets: Does it work? *Management Science*, 54(11):1935–1950.

Zhitlukhin, M. (2023a). Asymptotic minimization of expected time to reach a large wealth level in an asset market game. *Stochastics*, 95(1):67–78.

Zhitlukhin, M. (2023b). Capital growth and survival strategies in a market with endogenous prices. *SIAM Journal on Financial Mathematics*, 14(3):812–837.