Notes and examples: Buckingham Π theorem

PX391- S C Chapman

Universality- 1 d.o.f.

Pendulum

$$F = mg, F_t = mg\sin\theta, a_t = l\frac{d^2\theta}{dt^2}$$

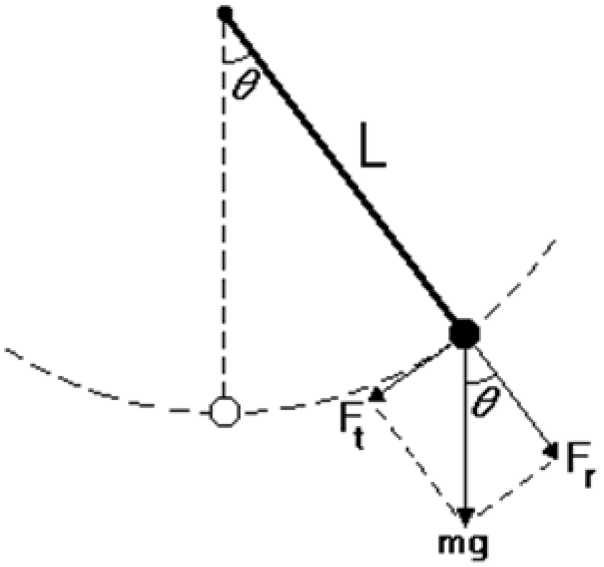
$$F_t = ma_t; \frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta = -\omega^2\frac{\partial V}{\partial\theta}$$

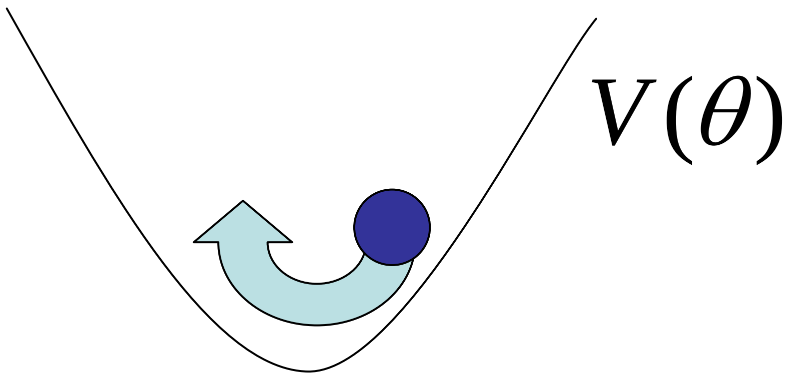
$$V(\theta) = 1 - \cos(\theta) \sim \frac{\theta^2}{2} + \dots$$

same behaviour at

any local minimum in $V(\theta)$

(insensetive to details)





Buckingham π theorem

System described by $F(Q_1 ... Q_p)$ where $Q_{1..p}$ are the relevant macroscopic variables

F must be a function of dimensionless groups $\pi_{1..M}(Q_{1..p})$

if there are R physical dimensions (mass, length, time etc.)

there are $M = P - R$ distinct dimensionless groups.

Then $F(\pi_{1..M}) = C$ is the general solution for this universality class.

To proceed further we need to make some intelligent guesses for $F(\pi_{1..M})$

See e.g. *Barenblatt, Scaling, self - similarity and intermediate asymptotics, CUP, [1996]*

also *Longair, Theoretical concepts in physics, Chap 8, CUP [2003]*

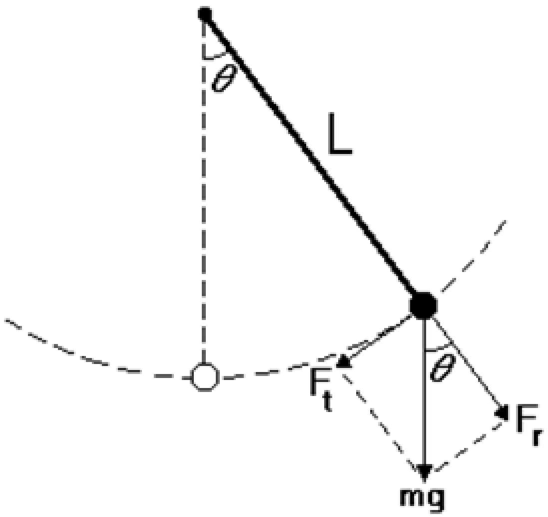
Example: simple (nonlinear) pendulum

System described by $F(Q_1 ... Q_p)$ where Q_k is a macroscopic variable

F must be a function of dimensionless groups $\pi_{1..M}(Q_{1..p})$

if there are R physical dimensions (mass, length, time etc.) there are $M = P - R$ dimensionless groups

Step 1: write down the relevant macroscopic variables:



variable	dimension	description
θ_0	$-$	angle of release
m	$[M]$	mass of bob
τ	$[T]$	period of pendulum
g	$[L][T]^{-2}$	gravitational acceleration
l	$[L]$	length of pendulum

Step 2: form dimensionless groups: $P = 5, R = 3$ so $M = 2$

$\pi_1 = \theta_0, \pi_2 = \dfrac{\tau^2 l}{g}$ and no dimensionless group can contain m

then solution is $F(\theta_0, \tau^2 l / g) = C$

Step 3: make some simplifying assumption: $f(\pi_1) = \pi_2$ then the period: $\tau = f(\theta_0)\sqrt{l/g}$

NB $f(\theta_0)$ is universal ie same for all pendula-

we can find it knowing some other property eg conservation of energy..

Example: fluid turbulence, the Kolmogorov '5/3 power spectrum'

System described by $F(Q_1 ... Q_p)$ where Q_k is a macroscopic variable

F must be a function of dimensionless groups $\pi_{1..M}(Q_{1..p})$

if there are R physical dimensions (mass, length, time etc.) there are $M = P - R$ dimensionless groups

Step 1: write down the relevant variables (incompressible so energy/mass):

variable	dimension	description
$E(k)$	$[L]^3[T]^{-2}$	energy/unit wave no.
ε_0	$[L]^2[T]^{-3}$	rate of energy input
k	$[L]^{-1}$	wavenumber

Step 2: form dimensionless groups: $P = 3, R = 2,$ so $M = 1$

$$\pi_1 = \frac{E^3(k)k^5}{\varepsilon_0^2}$$

Step 3: make some simplifying assumption:

$F(\pi_1) = \pi_1 = C$ where C is a non universal constant, then: $E(k) \sim \varepsilon_0^{2/3} k^{-5/3}$

Buchingham π theorem (similarity analysis)

universal scaling, anomalous scaling

System described by $F(Q_1 \ldots Q_p)$ where Q_k is a relevant macroscopic variable

F must be a function of dimensionless groups $\pi_{1..M}(Q_{1..p})$

if there are R physical dimensions (mass, length, time etc.) there are $M = P - R$ dimensionless groups

Turbulence:

variable	dimension	description
$E(k)$	$[L]^3[T]^{-2}$	energy/unit wave no.
ε_0	$[L]^2[T]^{-3}$	rate of energy input
k	$[L]^{-1}$	wavenumber

$$M = 1, \pi_1 = \frac{E^3(k)k^5}{\varepsilon_0^2}, E(k) \sim \varepsilon_0^{2/3} k^{-5/3}$$

introduce another characteristic speed....

variable	dimension	*description*
$E(k)$	$[L]^3[T]^{-2}$	energy/unit wave no.
ε_0	$[L]^2[T]^{-3}$	rate of energy input
k	$[L]^{-1}$	wavenumber
v	$[L][T]^{-1}$	characteristic speed

$$M = 2, \pi_1 = \frac{E^3(k)k^5}{\varepsilon_0^2}, \pi_2 = \frac{v^2}{Ek} \text{ let } \pi_1 \sim \pi_2^{\alpha}, E(k) \sim k^{-(5+\alpha)/(3+\alpha)}$$

Homogeneous Isotropic Turbulence and Reynolds Number

Step 1: write down the relevant variables:

variable	dimension	description
L_0	$[L]$	driving scale
η	$[L]$	dissipation scale
U	$[L][T]^{-1}$	bulk (driving) flow speed
ν	$[L]^2[T]^{-1}$	viscosity

Step 2: form dimensionless groups: $P = 4, R = 2,$ so $M = 2$

$\pi_1 = \frac{UL_0}{\nu} = R_E, \pi_2 = \frac{L_0}{\eta}$ and importantly $\frac{L_0}{\eta} = f(N),$ where N is no. of d.o.f

Step 3: d.o.f from scaling ie $f(N) \sim N^\alpha$ here $\frac{L_0}{\eta} \sim N^3$, or $N^{3\beta}$ or $\frac{L_0}{\eta} \sim \lambda^{N/3}$ or ...

Step 4: assume steady state and conservation of the dynamical quantity, here energy...

transfer rate $\varepsilon_r \sim \frac{u_r^3}{r}$, injection rate $\varepsilon_{inj} \sim \frac{U^3}{L_0}$, dissipation rate $\varepsilon_{diss} \sim \frac{\nu^3}{\eta^4}$ - gives $\varepsilon_{inj} \sim \varepsilon_r \sim \varepsilon_{diss}$

this relates π_1 to π_2 giving: $R_E = \frac{UL_0}{\nu} \sim \left(\frac{L_0}{\eta}\right)^{4/3} \sim N^\alpha, \alpha > 0$ thus N grows with R_E