Lecture 10 - Disordered systems: diagrammatics

The Anderson model of disorder

$$\hat{H} = \sum_{\langle ij \rangle} \varepsilon_{ij} c_i^\dagger c_j \; ; \qquad \varepsilon_{ij} = t \rightarrow \text{hopping amplitude}$$

ε_{ii} is a random quantity

$$\Updownarrow$$

$$\hat{H} = \underset{\text{free electrons}}{\hat{H}_0} + \underset{\text{disorder potential}}{\hat{V}(\underline{r})}$$

$\hat{V}(\underline{r}) = \sum_i V_i(\underline{r})$ — sum of impurity potentials over all impurities

or alternatively $V(r)$ is represented by a distribution

$$\mathcal{P}[V(\underline{r})],$$

which is normally chosen Gaussian,

$$\mathcal{P}[V(\underline{r})] = \frac{1}{Z} \exp\Big[-A \int V(\underline{r}) \underbrace{K(\underline{r}-\underline{r}')}_{\text{potential correlations}} V(\underline{r}')\, d\underline{r}\, d\underline{r}'\Big]$$

The simplest choice is a short-range potential,

$$K = \delta(\underline{r} - \underline{r}')$$

$$\langle V(\underline{r}) \rangle = 0$$

$$\langle V(\underline{r}) V(\underline{r}') \rangle = \frac{1}{2A} \delta(\underline{r}-\underline{r}') \equiv \frac{1}{2\pi \nu \tau} \delta(\underline{r}-\underline{r}')$$

ν: DoS; τ: it happens to be mean free time

The FT model is

$$S^{\pm} = \int \bar{\psi} \left(\varepsilon^{\pm} - \hat{H}\right) \psi \, d\underline{r}$$

Here $\varepsilon^{\pm} = \varepsilon \pm i\delta$; as the scattering is elastic, electron's energy is conserved, so that there are no integrals over time (or frequency in FT integrals).

The retarded (+) and advanced (−) GFs are

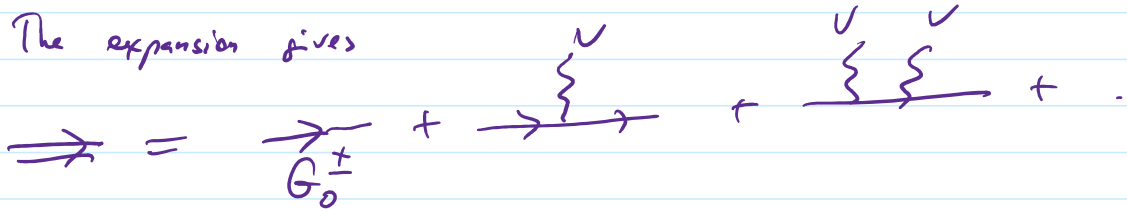
$$G^{\pm} = -i \frac{\int \psi \bar{\psi}\, e^{iS^{\pm}} \mathcal{D}\bar{\psi}\, \mathcal{D}\psi}{\underbrace{\int e^{iS^{\pm}} \mathcal{D}\bar{\psi}\, \mathcal{D}\psi}_{Z}}$$

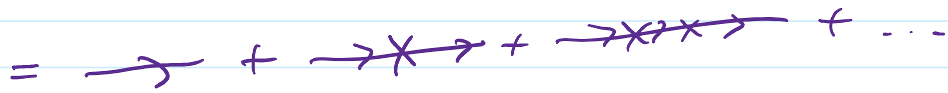
As $\hat{H} = \hat{H}_0 + \hat{V}$, we can naively expand

$$e^{iS^{\pm}} = e^{iS_0^{\pm} + iS_{imp}} = \sum \frac{(iS_{imp})^n}{n!} e^{iS_0^{\pm}}$$

$$S_{imp} = -\int \bar{\psi}\, \hat{V} \psi\, d\underline{r}$$

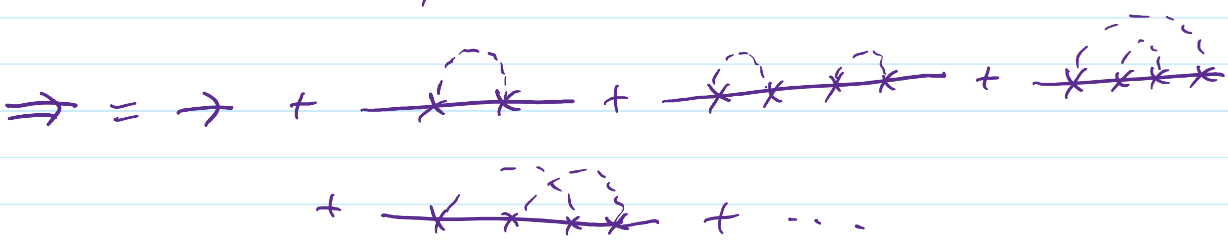
The expansion gives



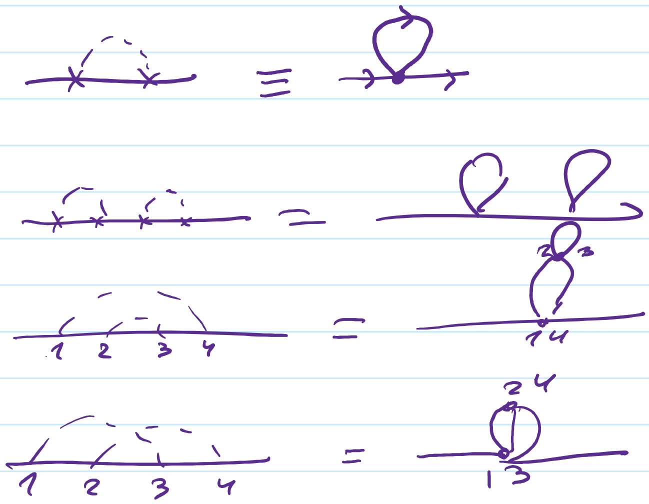


The aim is statistical properties after averaging over all realisations of random potential.

As $\langle V \rangle = 0$, we should have only even powers of V.



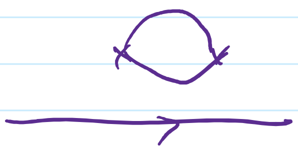
where $\times - - - \times \equiv \frac{1}{2\pi\nu\tau} \delta(\underline{r} - \underline{r}')$;



each term in the expansion is $\bar{\psi} V \psi$

then $(\bar{\psi} V \psi)(\bar{\psi} V \psi)$

If we expanded with all $(\bar{\psi} V \psi)$, we would also have



$\longrightarrow$ no physical meaning.

This diagram has no physical meaning, and we later see how to cancel it formally.

Naively, we can simply limit our considerations to diagrams with all crosses on a single line.

G_0 is the free electron f-n

$$G_0^{\pm}(\underline{p}, \varepsilon) = \frac{1}{\frac{p^2}{2m} - \varepsilon \pm i\delta} \equiv \frac{1}{\frac{p^2}{2m} - \underbrace{\frac{p_F^2}{2m}}_{\varepsilon} - \omega \pm i\delta}$$

ε is counted from ε_F

$$= \frac{1}{\xi - \varepsilon \pm i\delta}$$

From now on, we assume weak disorder

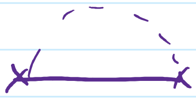
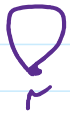
$\ell \gg \lambda_F \Leftrightarrow \varepsilon_F \gg \frac{\hbar}{\tau}$ $\quad\big|\quad \hbar = 1$

mean free path

$\ell \gg \frac{\hbar}{p_F} \Leftrightarrow p_F \ell \gg 1$

$$\varepsilon_F = \frac{v_F p_F}{2} \; ; \quad \varepsilon_F \tau = \frac{v_F \tau p_F}{2} = \frac{\ell p_F}{2}.$$

$\Rightarrow \; = \; \rightarrow \; +$  $+$ 2nd order ...

 $\equiv$  $= \int \frac{d^d p}{(2\pi)^d} G(p)$

$$= \nu \int_{-\infty}^{\varepsilon_F} d\xi \frac{1}{\varepsilon - \xi \pm i\delta} \approx \nu \int_{-\infty}^{\infty} \frac{d\xi}{\varepsilon - \xi \pm i\delta}$$

$\rightarrow$ the real part is $\rightarrow \infty$.

All divergences like that can be absorbed into $\mu = \varepsilon_F$, as we use grand canonical ensemble for a system with fixed N, by $N = -\frac{\partial \Omega}{\partial \mu}$ at the end.

We only need the Im of this diagram.

$$\int \frac{d\varepsilon}{\varepsilon \pm i\delta} \equiv \fint \frac{d\varepsilon}{\varepsilon} \mp \int 2\pi i \, \delta(\varepsilon)$$

$\int \frac{d\varepsilon}{\varepsilon + i\delta}$: choose the contour

$\int \frac{d\varepsilon}{\varepsilon + i\delta} =$  $= \underbrace{\lim_{\delta \to +0} \left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \right) \frac{1}{\varepsilon}}_{\equiv \fint \frac{d\varepsilon}{\varepsilon}}$

$\varepsilon = \delta e^{i\varphi}$ $\quad + \int_{\pi}^{0} \frac{\delta e^{i\varphi} d\varphi}{\delta} = 2\pi i$

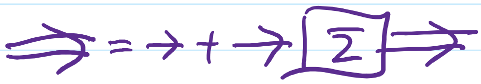
Hence,  $\equiv \Sigma$,

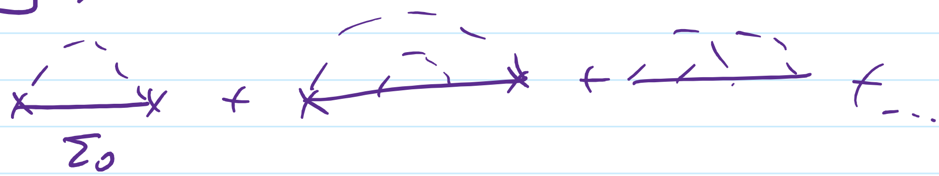
$$\operatorname{Im} \Sigma^{\pm} = \mp \frac{i}{2\tau}$$

$$\frac{1}{\xi \pm i\delta} = \frac{\xi \mp i\delta}{\xi^2 + \delta^2} \; ; \quad \operatorname{Re} \frac{1}{\xi \pm i\delta} = \frac{1}{\xi}$$

$$\operatorname{Im} (\;) = \frac{\delta}{\xi^2 + \delta^2}$$

Let's write Dyson eq for $G^{\pm}$:



self-energy $\boxed{\Sigma} =$ 

With Σ_0 only, we find

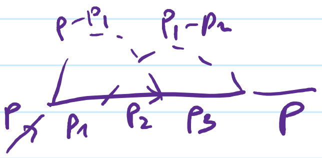
$$G = G_0 + G_0 \Sigma G_0 + G_0 \Sigma G_0 \Sigma G_0 \; . \; \ldots$$

$$= G_0 + G_0 \Sigma G$$

$$\Rightarrow \quad G^{\pm} = \frac{1}{G_0^{-1} - \Sigma} = \frac{1}{\varepsilon - \xi \pm i\delta \pm \frac{i}{2\tau}}$$

$$\rightarrow \frac{1}{\varepsilon - \xi \pm \frac{i}{2\tau}}$$

The higher order corrections to Σ:



$\sim \int (G)^3 \, dp_1 \, dp_2$ as $p_3 = p - p_1 + p_2$

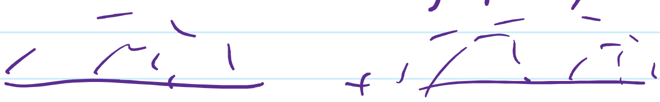
The poles are all in the same half-plane.

If $\int_{-\infty}^{\varepsilon_F} \rightarrow \int_{-\infty}^{\infty}$, it vanishes.

If $\int_{-\infty}^{\varepsilon_F} \to \int_{-\infty}^{\infty}$, it vanishes.

Keeping correction from $\int_{\varepsilon_F}^{\infty}$, we see that by dimension analysis that the result $\sim \frac{1}{(\varepsilon_F \tau)} << 1$.

(Note that all non-crossing parts, like



contribute but don't change the result, as

$$\mathrm{Im} \int \frac{1}{\varepsilon - \xi + \frac{i}{2\tau}} = -\frac{1}{2\tau}$$

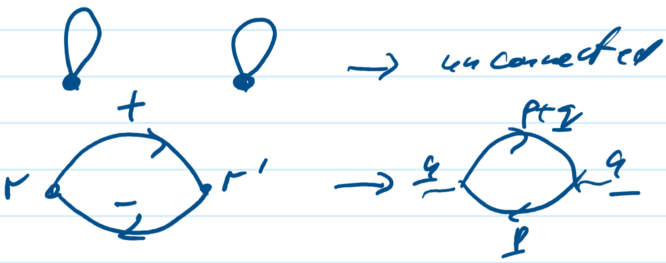
).

The GF defines the DoS, and the disorder makes no impact:

Disorder matters in correlations, in particular density-density, or current.

$$K(\underline{r},\underline{r}') \equiv \langle \rho(\underline{r})\, \rho(\underline{r}') \rangle \sim \langle \bar{\psi}(\underline{r}) \psi(\underline{r}) \bar{\psi}(\underline{r}') \psi(\underline{r}') \rangle$$

in GF, it has two contributions.



Hence, the lowest contribution to $K(\underline{r}, r')$, is

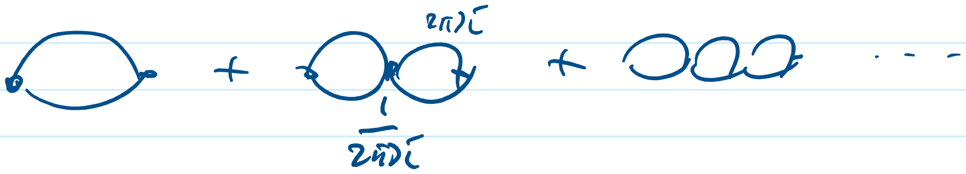
$$\sim \underbrace{\int \frac{d^d p}{(2\pi)^d}}_{\Rightarrow \nu \int d\xi} \frac{1}{\varepsilon - \xi_p + \frac{i}{2\tau}} \frac{1}{\varepsilon - \xi_{\underline{p}+\underline{q}} + \omega - \frac{i}{2\tau}}$$

If $\underline{q} = 0$ and $\omega = 0$, then

$$= \nu \int d\xi \; \frac{1}{\varepsilon - \xi + i/2\tau} \; \frac{1}{\varepsilon - \xi - i/2\tau}$$

$$= \nu \cdot 2\pi i \; \frac{1}{\frac{i}{\tau}} = 2\pi\nu\tau$$

The full correlation f-n would be

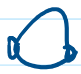


$$\equiv 2\pi\nu\tau \, \underbrace{(1 + 1 + 1 \ldots)} \to \text{divergence.}$$

Keeping $\xi_{\underline{p}+\underline{q}} = \frac{(\underline{p}+\underline{q})^2}{2m} - p_F^2 = \frac{\underline{p} \cdot \underline{q}}{m} + \frac{q^2}{2m} \approx \frac{\underline{p} \cdot \underline{q}}{m}$,

as it's divergent when $q = 0$, we only keep small q.

$\leftarrow$ angular integral

Hence, 

$$= \nu \int d\xi \int \frac{d\Omega}{\Omega_d} \; \frac{1}{\xi + \underline{v} \cdot \underline{q} + \omega + i/2\tau} \; \frac{1}{\xi - i/2\tau}$$

$$= 2\pi i \nu \int \frac{d\Omega}{\Omega_d} \; \frac{1}{\frac{i}{\tau} + \underline{v} \cdot \underline{q} + \omega}$$

$$= 2\pi\nu\tau \int \frac{d\Omega}{\Omega_d} \; \frac{1}{1 - i\tau \, \underline{v} \cdot \underline{q} - i\omega\tau}$$

expand in small ω & $\underline{q}$:

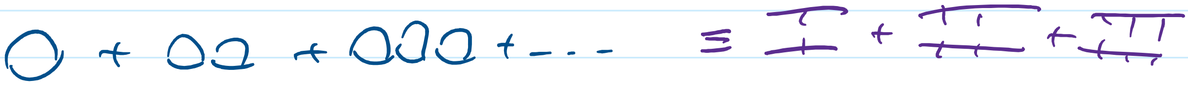
$$= 2\pi\nu\tau \left[1 + i\omega\tau - \overline{\tau^2 (v \cdot q)^2} \right]$$

averaging over directions of q

$$= 2\pi\nu\tau \left(1 + i\omega\tau - \frac{1}{d} \tau^2 v_F^2 q^2 \right)$$

$$D = \frac{v_F^2 \tau}{d} \equiv \frac{v_F l}{d}$$ — diffusion coefficient.

Hence,



$$= 2\pi\nu\tau \sum_{n=0}^{\infty} \left[1 + i\omega\tau - Dq^2\tau\right]^n$$

$$= \frac{2\pi\nu\tau}{(Dq^2 - i\omega)\tau} = \frac{2\pi\nu}{Dq^2 - i\omega} \equiv \text{diffuson}$$

It's divergent in low-frequency long-range limit, and hence describes long-range correlations in $d \leq 2$.

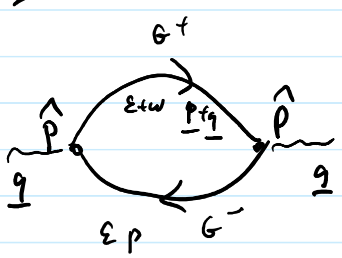
The "most important" observable is conductance.

Kubo formula:

Conductance $\sim \langle j \, j \rangle$

$$\hat{j} = \frac{ie}{\hbar m}\left[\psi \nabla \bar{\psi} - (\nabla \bar{\psi})\psi\right]$$

$\langle jj \rangle$ in the momentum representation:



The static limit $\omega, q \to 0$

The loop is similar to that in density-density but includes the angular average

$$\frac{e^2}{m^2}\langle \vec{p}\cdot(\vec{p}+\vec{q})\rangle \qquad ; \qquad \langle \vec{p}\cdot\vec{q}\rangle_{\text{angles}} \to 0$$

$$= \frac{e^2}{m^2}\langle \vec{p}^{\,2}\rangle = e^2 v_F^2$$

Otherwise, the single loop, as before, gives $2\pi\nu\tau$

The result is (in 2D)

$$\frac{1}{(2\pi)^2}\ \frac{e^2 v_F^2}{2}\left(2\pi\nu\tau\right)\underbrace{(2\pi)}_{\text{ang integration}} = e^2\,\frac{v_F^2\tau}{2}\,\nu = e^2\nu D$$

$$\equiv \frac{ne^2\tau}{m}$$

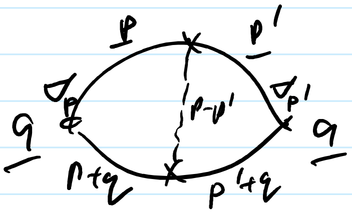
This is the Drude for static conductivity.

If $d\neq 2$, the result is

$$e^2\nu D\cdot L^{d-2} \equiv \underset{\substack{\uparrow\\ \text{conductance}}}{g}\, L^{d-2}\,\frac{e^2}{\hbar}$$

for a sample of L^d dimensions.

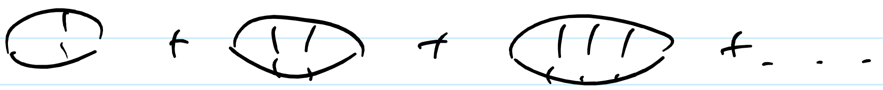
Corrections?



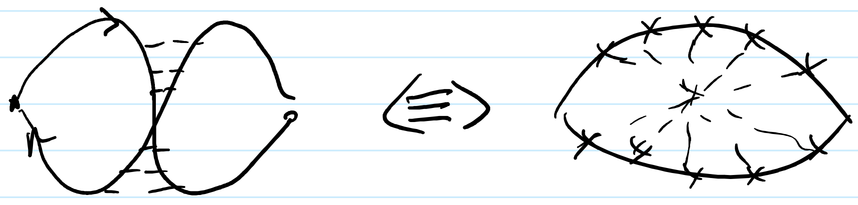
p & p' are uncorrelated,

hence, the angular averaging kills this contribution

However, although the ladder like before,

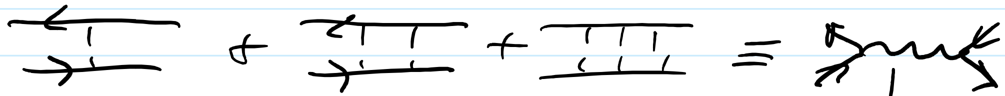


is not possible, a more complicated one exists:



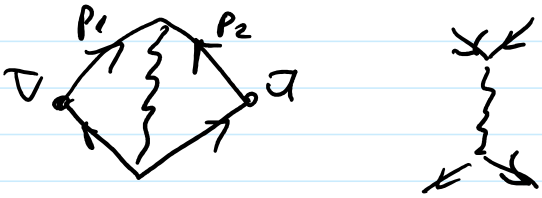
fan

Let's write the diffuson (as before) like



$$\frac{2\pi\nu}{Dq^2 - i\omega}$$

In a similar way, the sum of "fan" diagrams can be drawn as

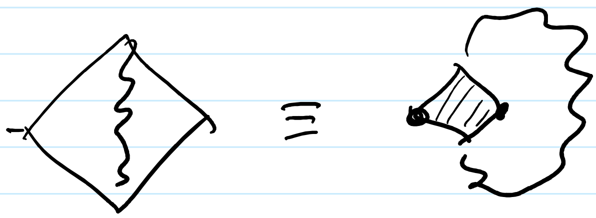


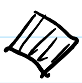
$\longrightarrow$ Cooperon

$|p_1 + p_2| \ll \frac{\hbar}{\ell}$

Diffuson reflect singularity at small transferred momenta & frequencies

Cooperon – singularity at small "sum" momenta & frequencies



 does not have divergence $\Rightarrow$ all momenta are close to $p_F \to \pm p_F$.

 $= \int G^4 d\xi \sim \nu\tau^3$

Divergence at small q, ω is in the contribution of Cooperon

$$\sim \int \frac{d^dq}{Dq^2 - i\omega} \longrightarrow \text{divergent for } d \leq 2$$

In particular, for $d=2$ and $\omega=0$

it's equal to $\frac{1}{D} \ln \frac{L}{\ell}$ $\rightarrow$ this comes

from cutting off momenta with $q > \frac{\hbar}{\ell}$

and with $q < \frac{\hbar}{L}$.

Combining Δ & ξ we find the correction

$$= -\ln \frac{L}{\ell}:$$

Dimensionless conductance

$$g = g_0 - \ln L/\ell .$$

2D: $g_0 \sim \nu D = \frac{m}{2\pi} \frac{v_F^2 \tau}{2} \sim \varepsilon_F \tau \gg 1$

We have a hint that something happens at

$$\ln \frac{L}{\ell} \sim g_0 \Rightarrow L \sim \ell e^{g_0} \equiv \ell_{loc}$$

At this scale, $g \to 0$ due to localisation — a statement not (yet) supported by calculations