Lecture 11 - Replica method. Nonlinear σ model.

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$$\hat{H} = \hat{H}_0 + V(\underline{r})$$

$$\mathcal{P}[V(\underline{r})] \sim \exp\left(-\pi\nu\tau \int d^d r\, V^2(\underline{r})\right)$$

$$\langle V(r)V(r')\rangle = \frac{1}{2\pi\nu\tau}\,\delta(\underline{r}-\underline{r}') . \quad \text{(white noise Gaussian)}$$

$$G^{\pm} = -i\,\frac{\int \psi\bar{\psi}\, e^{iS^{\pm}}\,\mathcal{D}\bar{\psi}\mathcal{D}\psi}{\int e^{iS^{\pm}}\,\mathcal{D}\bar{\psi}\mathcal{D}\psi}$$

$$S^{\pm} = \int \bar{\psi}(\varepsilon^{\pm} - \hat{H})\psi\, d\underline{r}$$

Typically, we need correlations that include two GFs

The task: to perform averaging ~~over~~ over $\mathcal{P}(V)$ before anything else.

$$\langle G\rangle = \int G\,\mathcal{P}(V)\,\mathcal{D}V$$

$$= \text{"} \int \frac{\int \psi\bar{\psi}\, e^{iS}\,\mathcal{D}\bar{\psi}\mathcal{D}\psi}{Z}\,\mathcal{P}(V)\,\mathcal{D}V$$

it's averaging over all realisations of V

Denominator is not allowing to change order of functional integration.

Mind you — supersymmetry gives a way to average

Instead — replicas.

We take $\psi \to \psi_i$ with N identical components, $i = 1, \dots, N$

Then,

$$\langle G \rangle = \int \frac{\int \psi_1(r)\bar{\psi}_1(r')\, e^{iS}\, D\bar{\psi}D\psi}{Z_1^N} P(V)\, DV$$

$$= \frac{1}{N} \int \frac{\int \underline{\underline{\psi}}\, \underline{\underline{\psi}}^{\dagger}\, e^{iS}\, D\bar{\psi}D\psi}{Z_1^N} P(V)\, DV$$

Here $S^{\pm} = \sum_{i=1}^{N} \int \bar{\psi}_i \left(\varepsilon^{\pm} - \hat{H}\right)\psi_i \, dr$

$$= \int \underline{\underline{\bar{\psi}}} \left(\varepsilon^{\pm} - \hat{H}\right) \underline{\underline{\psi}}\, dr$$

where $\underline{\underline{\psi}} = (\psi_1, \psi_2 \ldots \psi_N)^T$

In the limit $N \to 0$, we have $Z^N = 1$.

Note also that, since correlation f-ns can be found by differentiating $\ln Z$, we use

$$\ln Z_1 = \lim_{N\to 0} \frac{Z_1^N - 1}{N}$$

To get something nontrivial, we need products of $G^+ G^-$.

This requires to double the set of replicas;

$$\underline{\underline{\psi}} = (\psi_1^+ \ldots \psi_N^+, \psi_1^-, \ldots \psi_N^-)^T$$

To find $\langle G^+ G^- \rangle$, we introduce the source field. We have used source fields before, by including $-\bar{h}\psi - \bar{\psi}h$ into the action, and writing

$$\frac{\delta^2}{\delta h(r)\, \delta \bar{h}(r')}$$ to get $\bar{\psi}(r')\psi(r)$ in the pre-exponent.

Here, we can couple the source field to a pair of $\bar{\psi}^+ \psi^-$, adding to S the term

$$\int \underline{\underline{\bar{\Psi}}}(\underline{r})\, \underline{\underline{\eta}}\, \underline{\underline{\Psi}}(\underline{r})\, d\underline{r}$$

$\underline{\underline{\eta}}$ is $2N \times 2N$ matrix, where all $\eta^{++} = \eta^{--} = 0$, and $\eta^{\pm} = \eta^{\mp}$ are arbitrary $N \times N$ matrices

$$K_\omega(\underline{r}, \underline{r}') = \lim \frac{1}{N^2} \frac{\delta^2 Z(\eta)}{\delta \underline{\underline{\eta}}(\underline{r})\, \delta \underline{\underline{\bar{\eta}}}(\underline{r}')}$$

To take account of us dealing with products G^+G^-, we can rewrite

$$S^+ = \int \underline{\underline{\bar{\Psi}}}(\underline{r}) \left[\frac{\omega^+}{2} \Lambda - \hat{\xi} - V(\underline{r}) + \underline{\underline{\eta}} \right] \underline{\underline{\Psi}}(\underline{r})\, d\underline{r}$$

$$\Lambda = \operatorname{diag}(\underbrace{1 \dots 1}_{N}, \underbrace{-1 \dots -1}_{N})$$

and with this action we calculate

$$\langle G^+(\varepsilon + \tfrac{\omega}{2})\, G^-(\varepsilon - \tfrac{\omega}{2}) \rangle$$

As we assumed that the denominator doesn't matter, we write

$$\langle G^+ G^- \rangle = \frac{1}{N^2} \frac{\int \mathcal{D}V \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS}\, \mathcal{P}(V)\, \psi^+ \bar{\psi}^+ \psi^- \bar{\psi}^-}{\int \mathcal{D}V \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS}\, \mathcal{P}(V)}$$

We put the preexponent into S with the help of $\underline{\underline{\eta}}$, and then both integrals over $\mathcal{D}V$ and $\mathcal{D}\bar{\psi}\mathcal{D}\psi$ are Gaussian.

First, perform the Gaussian integration with

$$\mathcal{P}(V) = \int e^{-\pi\nu\tau \int dr\, V^2(r)}\, \mathcal{D}V$$

The Gaussian:

$$\int \mathcal{D}V\, e^{-\pi\nu\tau \int V^2(r) + i \underline{\bar{\Psi}} V \underline{\Psi}\, d\underline{r}}$$

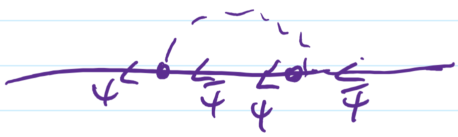
$$= \exp\left[- \frac{1}{4\pi\nu\tau} \int (\bar{\underline{\psi}}\,\underline{\psi})^2 \, d\underline{r} \right]$$

Hence, the effective action is

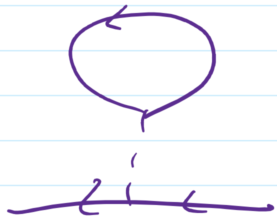
$$S(\underline{\eta}) = \int \Big\{ \bar{\underline{\psi}}(r) \left[-\hat{\xi} + \frac{\omega\Lambda}{2} + \underline{\eta} \right] \underline{\psi}(r) + \frac{i}{4\pi\nu\tau} (\bar{\underline{\psi}}\,\underline{\psi})^2 \Big\} \, d\underline{r}$$

Now we can justify (perturbatively) the replica trick:

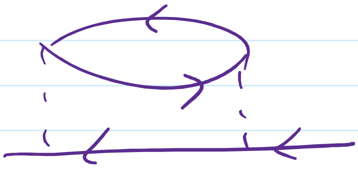
Expanding in $(\bar{\underline{\psi}}\,\underline{\psi})^2$, gives



but also terms like



or (in the next order)



Now $\underline{\psi}$ is a vector: any closed loop contains $\mathrm{Tr}\, \psi \bar{\underline{\psi}} = \sum \psi_i \bar{\psi}_i = N \psi_1 \bar{\psi}_1$.

As this is an extra N, in the replica limit its contribution vanishes.

Next, note that $(\bar{\underline{\psi}}\,\underline{\psi})^2 = \mp \mathrm{Tr}\, q^2$ (fermions: upper sign; bosons: lower sign)

where $\underline{\underline{q}} = \underline{\psi} \otimes \bar{\underline{\psi}}$ — a $2N \times 2N$ matrix.

Hence, there's a term $\frac{1}{4\pi\nu\tau} \mathrm{tr}\, \underline{q}^2$ in S.

Now – Hubbard-Stratonovich transformation.

We introduce (a matrix) field $\underline{Q}$ where the symmetry of $\underline{\underline{Q}}$ is the same of $\underline{\underline{q}}$.

Add to the numerator & denominator

$$\int \mathcal{D}Q\, e^{-a\,\mathrm{Tr}\,Q^2} \qquad (\mathrm{Tr} = \int d\underline{r}\cdot \mathrm{tr} \quad).$$

in such a way that the quartic in ψ term is decoupled:

$$e^{-\frac{1}{4\pi\nu\tau}\int(\bar{\psi}\psi)^2 d\underline{r}} = \frac{1}{Z}\int \mathcal{D}Q\, \exp(-F)$$

where

$$F = \int\left[-\frac{\pi\nu}{8\tau}\,\mathrm{tr}\,Q^2 + \frac{i}{2\tau}\,\bar{\psi}\,Q\,\psi\right]d\underline{r}$$

$$\underbrace{\qquad\qquad}_{-\frac{i}{2\tau}\,\mathrm{tr}\,q\,Q}$$

$$= -\frac{\pi\nu}{8\tau}\,\mathrm{Tr}\,Q^2 - \frac{i}{2\tau}\,\mathrm{Tr}\,q\,Q$$

making shift of $Q \to Q - \ldots q$, returns the previous expression after integrating over ψ.

As a result, we got rid of the quartic term $\sim \mathrm{Tr}\,q^2$ to get

$$F[Q] = \frac{\pi\nu}{8\tau}\,\mathrm{Tr}\left[Q^2 - \frac{1}{2}\ln\left(\frac{\omega}{2}\Lambda + \underline{\underline{\eta}} - \hat{\xi} + \frac{i}{2\tau}Q\right)\right]$$

Now the partition f-n $Z = \int \mathcal{D}Q\, \exp(-F)$.

The next step – saddle-point.

Assuming first that Q is a homogeneous (Σ-independent) field, calculate

$\frac{\partial F}{\partial Q} = 0$, neglecting $\underline{\eta}$ and using the static limit, $\omega = 0$

$$\mathrm{Tr}\, \frac{1}{-\xi + \frac{i}{2\tau} Q} = \nu Q$$

$$\parallel$$

$$\int d\underline{p}\, \frac{1}{-\xi + \frac{i}{2\tau} Q} = \int d\xi\, \frac{\xi + \frac{i}{2\tau} Q}{\xi^2 + \frac{Q^2}{4\tau^2}} = Q$$

Let's assume $Q^2 = 1$ ↑

$$\frac{iQ}{2\tau} \int \frac{d\xi}{\xi^2 + \frac{1}{4\tau^2}} = \underline{Q}$$

> i.e. the equation is satisfied with $Q^2 = \mathbb{1}$.

Now – corrections to the saddle-point:

two types: (i) $Q \to \mathbb{1} + \delta Q$

One can show that these corrections (longitudinal) $\sim \frac{1}{(\varepsilon_F \tau)^2}$.

(ii) $\underline{Q}^2 = \mathbb{1}$ but is Σ-dependent (transverse corrections).

Back to the expression for F:

$\underline{Q}^2 = \mathbb{1}$ is satisfied by any

$$\underline{Q} = \underline{U}^\dagger \underline{\Lambda}\, \underline{U} \Rightarrow \underline{Q}^2 = \underline{U}^\dagger \Lambda \underline{U}\, \underline{U}^\dagger \Lambda \underline{U} = \underline{U}^\dagger \Lambda^2 \underline{U} = \mathbb{1}$$

Note $\underline{\underline{Q}}^2 = \underline{\underline{Q}}\,\underline{\underline{Q}}^\dagger = \mathbb{1} \rightarrow \underline{\underline{Q}}$ is unitary.

$\mathrm{Tr}\,\underline{\underline{Q}} = 0$

$$F = -\frac{1}{2}\mathrm{Tr}\ln\left[-\hat{\xi} + \frac{i}{2\tau}Q + \frac{\omega}{2}\Lambda\right]$$

$$= -\frac{1}{2}\ln\det\left[\underline{\underline{U}}\left(-\hat{\xi} + \frac{i}{2\tau}U^\dagger\Lambda U + \frac{\omega}{2}\Lambda\right)U^\dagger\right]$$

$$= -\frac{1}{2}\mathrm{Tr}\ln\left[-U\hat{\xi}U^\dagger + \frac{i}{2\tau}\Lambda + \frac{\omega}{2}U\Lambda U^\dagger\right]$$

$$-U\xi U^\dagger + \hat{\xi} - \hat{\xi}$$

$$= -\frac{1}{2}\mathrm{Tr}\ln\left[\underbrace{-\hat{\xi} + \frac{i}{2\tau}\Lambda}_{\mathcal{G}_0^{-1}} - U[\xi, U^\dagger] + \frac{\omega}{2}U\Lambda U^\dagger\right]$$

$$\rightarrow -\frac{\omega}{4}\mathrm{Tr}\,\mathcal{G}_0 U\Lambda U^\dagger + \frac{1}{4}\mathrm{Tr}\,\mathcal{G}_0 U[\hat{\xi}, U^\dagger]G_0 U[\hat{\xi}, U^\dagger]$$

2nd order of exp ln

In the ω-term,

$$-\frac{\omega}{4}\mathrm{Tr}\,\mathcal{G}_0(\underline{r}, \underline{r}')\left(\Lambda U\Lambda U^\dagger\right)_{\underline{r}} = -\frac{\omega}{4}\mathcal{G}_0(\underline{r}, \underline{r})\,\mathrm{Tr}_r\,\Lambda U\Lambda U^\dagger_{\underline{r}}$$

$\Downarrow$

$\underline{r} = \underline{r}'$

$$\mathrm{Tr}\,\underbrace{U^\dagger\Lambda U}_{Q}\Lambda = \mathrm{Tr}\,\Lambda Q$$

$$\mathcal{G}_0(\underline{r}, \underline{r}) = \int \mathcal{G}_0(p)\,dp$$

$$= \nu\int d\xi\, G_0(\xi) = \nu\int d\xi\,\frac{\frac{i}{2\tau}\Lambda}{\xi^2 + \frac{1}{4\tau^2}}$$

$$= i 2\nu\pi\tau$$

Hence, the ω term is

$$-\frac{i\omega\pi\nu}{4}\mathrm{Tr}\,\Lambda Q$$

The 2nd term, $\sim \mathrm{Tr}\,G_0 U[\xi, U^\dagger]G_0 U[\xi, U^\dagger]$

Introduce $\underline{A} = \underline{U \nabla U^\dagger}$, using $[\hat{\xi}, U^\dagger] =$

$= v_F \underline{n} \cdot \nabla$
↑ unit vector

since $\hat{\xi} = -\frac{1}{2m}\nabla^2 - \varepsilon_F = \frac{\hat{p}^2}{2m} - \frac{p_F^2}{2m} = \frac{(\hat{p} - p_F)(p + p_F)}{2m}$
$= v_F \vec{n} \cdot \nabla$

Then, the trace above becomes

$$\frac{v_F^2}{4} \int d\underline{r}\, d\underline{r}' \left[\langle r | \mathcal{G}_0 | \underline{r}' \rangle (\underline{n} \cdot \underline{A})_{\underline{r}} \langle r' | \mathcal{G}_0 | r \rangle (n \cdot A)_{r'} \right]$$

$$= \frac{v_F^2}{4} \sum_{\underline{p}, \underline{q}} G_{0\,\underline{p}}\, A_{\underline{q}}\, G_{0,\,\underline{p+q}}\, A_{\underline{q}}$$

We keep the lowest-order terms in $\underline{q}$:

$$\frac{\nu v_F^2 \tau}{4} \sum_{\underline{q}} \nu \int d\xi \frac{(\xi + i\Lambda) A_{\underline{q}}}{\xi^2 + 1} \frac{(\xi + i\Lambda) A_{\underline{q}}}{(\xi^2 + 1)} d\xi$$

$$= \frac{\pi \nu D}{8} \sum_{\underline{q}} \mathrm{tr}\left[A_q, \Lambda\right]^2$$

$D = \frac{v_F^2 \tau}{2}$ — diffusion coef. in $d = 2$.

$$\mathrm{tr}\left[A_q, \Lambda\right]^2 = \mathrm{tr}\left[U \nabla U^\dagger, \Lambda\right]^2$$

$$= \mathrm{tr}\, U \nabla U^\dagger \Lambda U \nabla U^\dagger = \mathrm{tr}(\nabla Q)^2$$

since $\nabla Q = \nabla (U^\dagger \Lambda U) = \nabla U^\dagger \Lambda U + U^\dagger \Lambda \nabla U$.

To conclude, keeping only transverse fluctuations around the saddle point, we arrive at

$$F[Q;\omega] = \int d\underline{r}\ \mathrm{tr}\left[\frac{\pi\nu D}{8}(\nabla Q)^2 - \frac{i\pi\nu\omega}{4}\Lambda Q\right]$$

NLσM

where $\underline{Q}^2 = \mathbb{1}$

$$Q^2 = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}^2 = \begin{pmatrix} Q_{11}^2 + Q_{12}Q_{21} & \\ & \cdots \end{pmatrix} = \mathbb{1}$$

$$Q_{11} = \sqrt{\mathbb{1} - Q_{12}Q_{21}} = 1 - \frac{1}{2}Q_{12}Q_{21} + \cdots$$