Lecture 12 - Using σ model: perturbation theory and renormalisation group

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$$\hat{H} = \hat{H}_0 + \hat{V} \; ; \qquad \langle \hat{V}(\underline{r})\, V(\underline{r}')\rangle = \frac{1}{2\pi \nu \tau}\, \delta(\underline{r} - \underline{r}') ;$$

$$\Longleftrightarrow \qquad F[Q;\omega] = \mathrm{Tr}\left[\frac{\pi \nu D}{8}(\nabla Q)^2 - \underbrace{\frac{i\pi\nu\omega}{4}\Lambda Q}_{\text{symmetry-breaking}}\right]$$

Q is $2N \times 2N$ matrix obeying

$\underline{Q^2 = \mathbb{1}}$ and $\mathrm{Tr}\, Q = 0$; $N \to 0$ in the final expressions

$$\Rightarrow \quad Q = \underline{U^+} \Lambda U \quad \text{where} \quad \Lambda = \mathrm{diag}(\mathbb{1}_N, -\mathbb{1}_N)$$

Symmetries: $\qquad Q \sim \psi \times \bar{\psi}$

$U \in S(2N)$

global

$$S(2N) = \begin{cases} U(2N), & \text{in the presence of a weak magnetic field} \\ Sp(2N) - \text{symplectic, made of quarternions,} & \text{for the original model with } B=0; \\ O(2N) - \text{orthogonal,} & \text{when spin-orbit int. is there} \end{cases}$$

local: for $U \in S(2N)$ we choose subgroup $S(N) \otimes S(N)$,

e.g. $\qquad \tilde{U} = \begin{pmatrix} \tilde{U}_N & 0 \\ 0 & \tilde{\tilde{U}}_N \end{pmatrix}$

Let $\quad Q_0 = U_0^+ \Lambda U_0 ; \qquad U_0 \to \tilde{U} U_0$

$$Q_0 \to U_0^+ \tilde{U}^+ \Lambda \tilde{U} U_0 = U_0^+ \underbrace{\tilde{U}^+ \tilde{U}}_{\mathbb{1}} \Lambda U_0 = U_0^+ \Lambda U_0$$

$$\Rightarrow \quad \tilde{U} \to \tilde{U}(\underline{r}).$$

Q is said to belong to a homogeneous space

$$\frac{S(2N)}{S(N)\otimes S(N)}.$$

Let's choose representation by $U = e^{-\frac{W}{2}} \Rightarrow U^\dagger = e^{\frac{W}{2}}$

where $W = \begin{pmatrix} 0 & B \\ -B^\dagger & 0_N \end{pmatrix}$; $W^\dagger = -W$

$\{\Lambda, W\} = 0$; B is arbitrary $N\times N$ matrix

$$\Rightarrow Q = e^{\frac{W}{2}} \Lambda e^{-\frac{W}{2}} = \left(\mathbb{1} + W + \frac{W^2}{2} + \ldots\right)\Lambda$$

$$F_0 \equiv \mathrm{Tr}\,(\nabla Q)^2\Big|_{0^{th}\text{ order}} = \mathrm{Tr}\,\Lambda\nabla W \Lambda \nabla W = -\mathrm{Tr}\,(\nabla W)^2$$

$$= -\mathrm{Tr}\begin{pmatrix} 0 & \nabla B \\ -\nabla B^\dagger & 0 \end{pmatrix}^2 = 2\,\mathrm{Tr}\,\nabla B \nabla B^\dagger$$

B is a unconstrained $N\times N$ field.

$$\Rightarrow F_0[Q] = \frac{\pi\nu D}{4}\mathrm{Tr}\left[\nabla B \nabla B^\dagger - \frac{i\pi\nu\omega}{2} B B^\dagger\right]$$

In q-space,

$$F_0 = \pi\nu\,\mathrm{Tr}\; B_q \underbrace{(Dq^2 - i\omega)}_{\mathcal{G}_0^{-1}} B_{-q}$$

$$\langle B_{ij}(q)\, B^\dagger_{kl}(-q)\rangle = \frac{1}{2\pi\nu}\frac{1}{Dq^2 - i\omega}\,\delta_{ij}\cdot\delta_{kl}$$

only diffusons for unitary model

$(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$

diffusons + Cooperons for symplectic

$\delta_{ii} = N$. Diagonal term $\sim N^2$ to cancel

$\delta_{ii} = N$; Diagonal term $\sim N^2$ to cancel $\frac{1}{N^2}$ in all correlation f-ns,

e.g.

$$K_\omega(\underline{r}, \underline{r}') = \frac{1}{2N^2} \mathrm{Tr} \frac{\delta^2 Z(h)}{\delta h(r)\, \delta h(r')} \Bigg|_{\substack{h=0 \\ N=0}}$$

density-density

where h is a source field which enters like ω giving an additional term

$\mathrm{Tr}\, hQ$ in the σ-model.

Perturbation expansion :

$$Z = \langle e^{-F} \rangle = \int \mathcal{D}Q\; e^{-F[Q]}$$

without symmetry-breaking terms ($\sim\omega$ or $\sim h$)

$$F[Q] = \mathrm{Tr}(\nabla Q)^2$$

$$= \mathrm{Tr}\, \Lambda \left(\nabla W + \nabla \frac{W^2}{2} + \ldots\right) \Lambda \left(\nabla W + \nabla \frac{W^2}{2} + \ldots\right)$$

$$= \underbrace{-\mathrm{Tr}\, \nabla W \nabla W}_{F_0} + \text{higher order terms}$$

Adding the source field gives

$$\mathrm{Tr}\, hQ = \underline{\mathrm{Tr}\, hW} + \underline{\frac{hW^2}{2}} + \ldots$$

We need $\sim h^2$ term to calculate K.

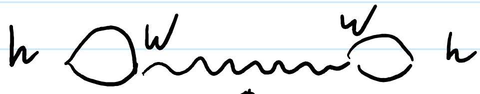
The lowest term is $\mathrm{Tr}\, hW$;

$$\int e^{-F} \mathcal{D}Q = \int e^{-F_0 - (F - F_0)} \mathcal{D}Q$$

$$= \int \sum \frac{(F_0 - F)^n}{n!}\, e^{-F_0}\, \mathcal{D}Q$$

$$= \left\langle \sum \frac{(F_0 - F)^n}{n!} \right\rangle$$

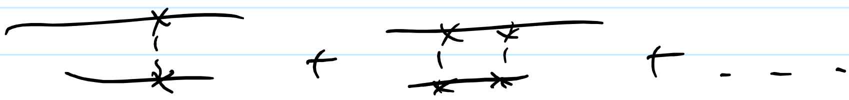
The lowest $\sim h^2$ term is $\mathrm{Tr}\, hW\, \mathrm{Tr}\, hW$

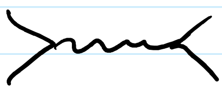
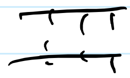


$$\langle \mathrm{Tr}\, hW\, \mathrm{Tr}\, hW \rangle \sim \mathrm{Tr}\, \frac{\pi\nu}{Dq^2 - i\omega}\, \mathrm{tr}\, h^2$$

$$\sim \ln\frac{(r-r')}{L} \; ; \qquad \frac{\delta^2\, \mathrm{tr}\, h^2}{\delta h\, \delta h'} \sim N^2$$

This is equivalent to the sum



so that  $= \sum$ 

The combination of matrix & functional averaging is encapsulated by

$$\langle W_{ij}(q) W_{kl}(q') \rangle = \frac{1}{2\pi\nu(Dq^2 - i\omega)}\, \delta(q+q')\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

$$\Downarrow$$

$$\langle \mathrm{Tr}\, WPWR \rangle = (\cdots)\left[\mathrm{Tr}\left(\Lambda P \Lambda R^{+} - P R^{+}\right) + \mathrm{Tr}\,\Lambda P\, \mathrm{Tr}\,\Lambda R - \mathrm{Tr}\, P\, \mathrm{Tr}\, R\right]$$

$$\langle \mathrm{Tr}\, WP \;\; \mathrm{Tr}\, WR \rangle = \ldots \mathrm{Tr}\big(\lambda P \Lambda R - \lambda P \Lambda R^{+} - PR + PR^{+}\big).$$

Conductance..

$$J_\alpha = \frac{i}{2}\left[\psi \otimes \partial_\alpha \bar{\psi} - \partial_\alpha \psi \otimes \bar{\psi} \right]$$

Kubo formula:

$$G_{\alpha\beta} = \frac{1}{L^2} \int K_{\alpha\beta}\, dr\, dr'$$

$$K_{\alpha\beta} = \frac{4}{\pi N^2} \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \; \mathrm{Tr}\, J_+ J_- e^{iS}}{Z}$$

where $J_\pm = C_\pm J$; $\quad C_\pm = \frac{1}{2}(1 \pm \sigma_3) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{cases}$

After the Hub-Str, we can write

$$K_{\alpha\beta} = \frac{1}{16\pi N^2} \mathrm{Tr} \frac{\delta^2 Z(A)}{\delta A_\alpha(r)\, \delta A_\beta},$$

where in the σ-model

$$\nabla_\alpha Q \to \underbrace{\partial_\alpha Q - [A_\alpha, Q]}_{\text{long derivative}}$$

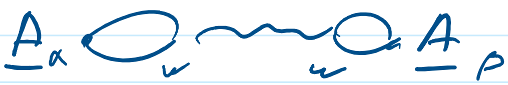
Perturbatively, we should keep $\sim A^2$ in the expansion $\sum \frac{(F_0 - F)^n}{n!}$

Can it be $\operatorname{Tr} A_\alpha W \operatorname{Tr} A_\beta W$?

Now as $A_{ext} = \begin{pmatrix} 0 & A \\ A^+ & 0 \end{pmatrix}$,

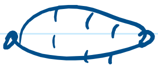
$\operatorname{Tr} WA = 0$.

If we had it, it would be



$= 0$ by angle averaging.

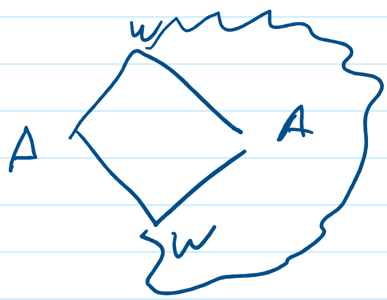
In diagrams,  $\rightarrow$ Drude, but

correction like  $\rightarrow 0$

Next term containing A^2 is

$$\frac{\pi\nu D}{2} \operatorname{Tr} W A_\alpha W A_\alpha$$

coming directly from $[A_\alpha, Q]$ contribution to $(\nabla Q)^2$.



$$\int \pi\nu D \frac{1}{\pi\nu(Dq^2 - i\omega)} d^d q \sim \ln \frac{L}{\ell}$$

In this correction, D is cancelled, which

gives $G_{\alpha\beta} \rightarrow G_{\alpha\beta}$ + log corrections.

It's diagonal when $\beta = 0$, and introducing

$g = G/(e^2/h)$, we find

$$g = g_0 - \ln L/\ell = g_0 (1 - g_0^{-1} \ln L/\ell)$$

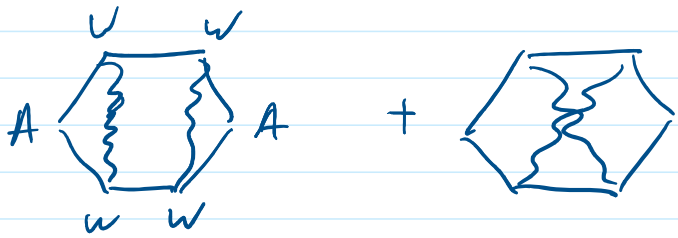
the lowest order; we might have

$$\sum_{n=1}^{\infty} (g_0^{-1} \ln L/\ell)^n$$

However, no contributions $\sim (g_0^{-1} \ln L/\ell)^n$ with $n > 1$ arises

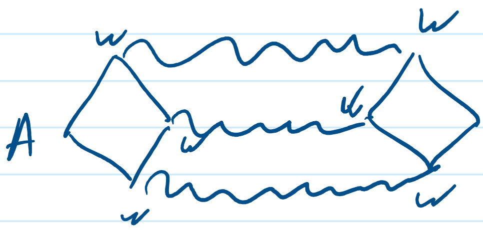
Expanding further in W, and keeping $\sim A^2$, we have

$$\langle \mathrm{Tr}\, A W^2 A W^2 \rangle \quad \text{and} \quad \langle \mathrm{Tr}\, A W^3\, \mathrm{Tr}\, A W^3 \rangle$$



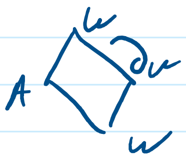
each $\sim g_0^{-2} \ln^2 L/\ell$

but so is



each $\sim \frac{1}{Dq^2}$

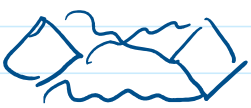
but



contains one ∂W

as $\nabla Q \to \partial_\mu Q - [A, Q]^2$

also



etc

The sum $=0$ $\longrightarrow$ the hint of renormalizability.

The RG, indeed, exists.

RG in the σ-model.

The idea – separating fast and slow variables in the symmetry-preserving way.

$$U \to \underset{\substack{\uparrow \\ \text{fast variables} \\ \lambda q < q < q_0}}{U_0(q)}\ \underset{\substack{\uparrow \\ \text{slow} \\ q < \lambda q}}{\tilde{U}(q)}$$

$$Q = \tilde{U}^+ \underbrace{U_0^+ \Lambda U_0}_{Q_0} \tilde{U} = \tilde{U}^+ \underset{\substack{\uparrow \\ \text{fast}}}{Q_0} \tilde{U} \quad ; \quad \tilde{Q} \equiv \tilde{U}^+ \Lambda \tilde{U}$$

The target: $\tilde{F}[Q]$ which is defined as

$$e^{-\tilde{F}(\tilde{Q})} = \int e^{-F[Q]} \mathcal{D} Q_0$$

$$F[Q] = \underset{\substack{\downarrow \\ \text{the same as } F[Q] \\ = Tr(\nabla \tilde{Q})^2}}{F(\tilde{Q})} + \underset{\substack{\text{fast} \\ \text{quadratic}}}{F_0[Q_0]} + \underset{\text{mixed}}{F[Q_0, \tilde{Q}]}$$

$$F_0 = -\frac{\pi \nu D}{8} Tr\left[(\nabla W_0)^2 + \underbrace{\lambda^2 q_0^2 W_0^2}_{\substack{\text{invariant cutoff} \\ \text{allows us to integrate} \\ \text{over all } q}}\right]$$

allows us to integrate over all q

In the mixed term, we have contributions that contain

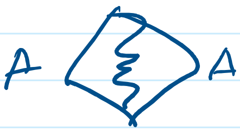
$$F^{(0)}(\tilde Q, Q_0) = \frac{\pi\nu D}{4}\,\mathrm{Tr}\,\Lambda A_\alpha \Lambda W [W, A_\alpha]$$

where $A_\alpha \equiv (\partial_\alpha \tilde U)\tilde U^\dagger = -\tilde U \partial_\alpha \tilde U^\dagger$

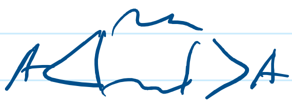
$$F^{(1)} = -\frac{\pi\nu D}{4}\,\mathrm{Tr}\left(\partial_\alpha W_0 [W_0, A_\alpha]\right)$$

Then we expand e^{-F} in these mixed terms and average with fast integration: this gives

$$\tilde F(\tilde Q) = \tilde{\mathcal F}(\tilde Q) + \langle F^{(0)}\rangle_0 - \frac{1}{2}\langle (F^{(1)})^2\rangle_0$$



$$\int \frac{d^d q}{q^2+\lambda^2 q_0^2}$$



$$\int \frac{q^2\, d^d q}{(q^2+\lambda^2 q_0^2)^2}$$

Averaging produces

$$\mathrm{Tr}\,\Lambda A_\alpha \Lambda A_\alpha \to \mathrm{Tr}\,\Lambda \partial_\alpha \tilde U \tilde U^\dagger \Lambda \partial_\alpha \tilde U \tilde U^\dagger$$
$$\to \mathrm{Tr}(\partial \tilde Q)^2 .$$

Re-exponentiating these terms, we find that (in $d=2$) $\frac{\pi\nu D}{2} \to \frac{\pi\nu D}{2} - \ln\lambda^{-1}$

that (in $d=2$) $\quad \frac{\pi V D}{8} \to \frac{\pi V D}{8} - \ln \lambda^{-1}$

$$\iff g \to g - \ln \lambda^{-1}$$

$$\iff \frac{dg}{d \ln \lambda^{-1}} = -1 + O\left(\frac{1}{g}\right) \equiv \beta(g)$$

this would lead to $g \simeq g_0 - g_0^{-1} \ln L/\ell$

$$- g_0^{-n} \ln L/\ell$$

where n is the 1^{st} non-vanishing higher-order term.