Lecture 3 - Diagrammatic summation. Dyson equation.

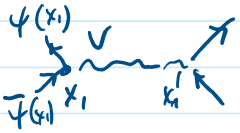
15 February 2023 14:00

Last time: $H = H_0 + H_{int}$ ← pair interaction

$$e^{i(S_0 + S_{int})} = \sum_{n=0}^{\infty} \frac{(iS_{int})^n}{n!}$$

$$G(x,x') = -\frac{i}{Z} \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\; \psi(x)\bar{\psi}(x') e^{iS_0} \sum \frac{(iS_{int})^n}{n!}$$

$$S_{int} = \frac{1}{2} \int \bar{\psi}(x_1)\bar{\psi}(x_1')\, V(x_1 - x_1')\, \psi(x_1')\psi(x_1)\, dx_1 dx_1'$$



In $S_{int}^n \cdot \psi\bar{\psi}$ we have $(2n+1)$ pairs of $\bar{\psi}\psi$.

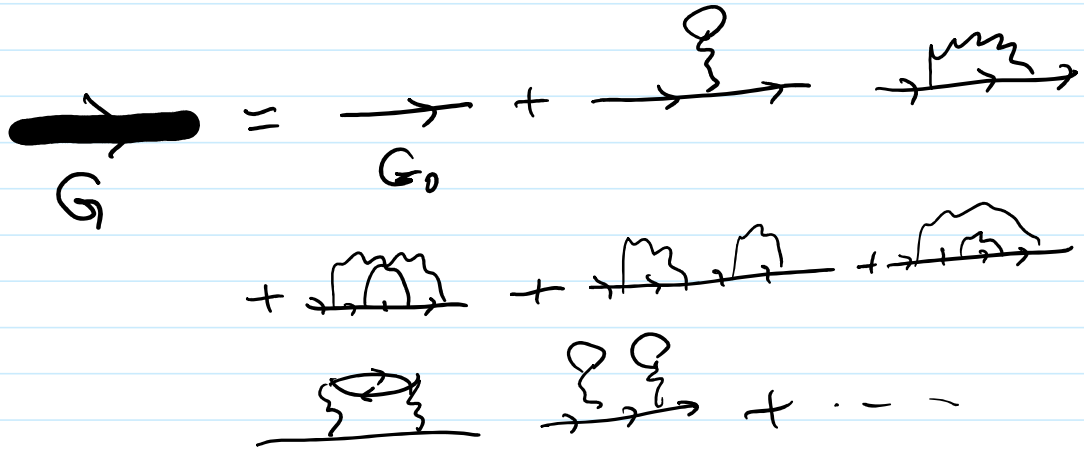
$n!$ and $\frac{1}{2^n}$ are cancelled by $n!$ permutations of all $\{x_i\}$, and by 2^n swaps of $x_i \leftrightarrow x_i'$.

In S^n we have i^n but each GF requires $(-i)$, as $G_0 = -i\langle \psi(x_j)\bar{\psi}(x_j') \rangle$

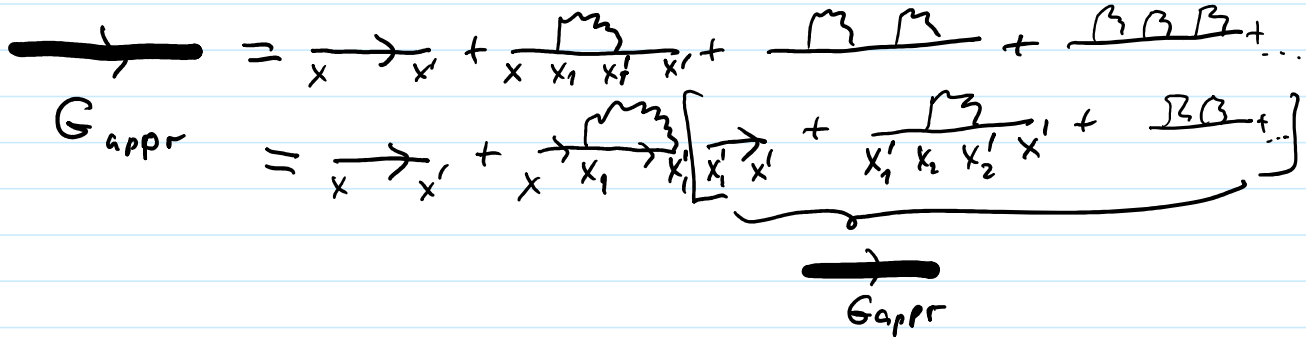
To get the correct power, we need to multiply n^{th} term by i^n.

We cancel all unconnected diagrams by contributions from $\frac{1}{Z}$.

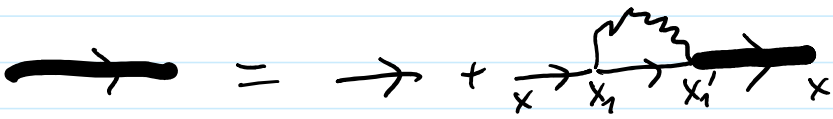
Using all possible connections, we find



Let's choose one class of diagrams:



Hence, we have eq:



$$\Leftrightarrow \quad G_{app}(x-x') = G_0(x-x') + \int G_0(x-x_1)\, G_0(x_1-x_1')\, G_{app}(x_1'-x')\, V(x_1-x_1')\, dx_1 dx_1'$$

If we make FT, $G_0(q) = \int G_0(x)\, e^{ipx}\, dx$

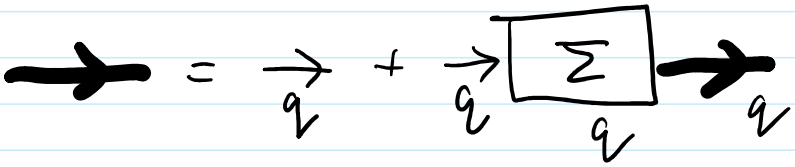
$\underline{p}, \omega$

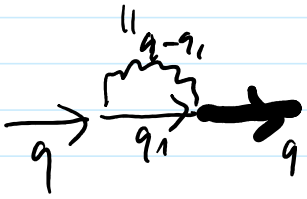
$$G_0(x) = \int G_0(q)\, e^{-ipx}\, \underbrace{\frac{d^{d+1}q}{(2\pi)^{d+1}}}_{\equiv dq}$$

The same for G_{app}.

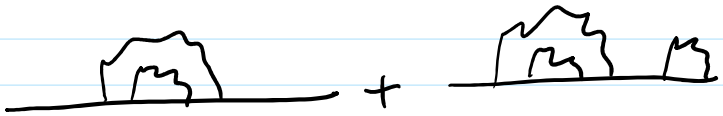
$$\Rightarrow \quad G_{app}(q) = G_0(q) + G_0(q)\,\Sigma_{app}(q)\,G(q)$$

where $\Sigma_{app}(q) = \int dq_1\, G_0(q_1)\, V(q - q_1)$

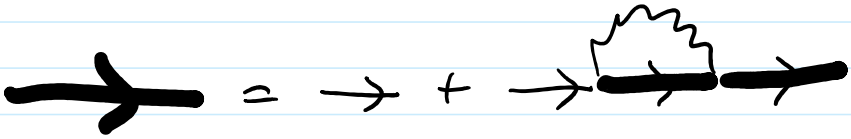




This unjustified app involves very limited class.

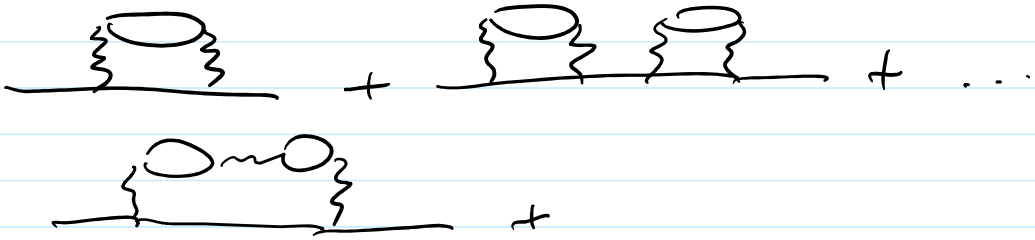
Let's add 

These can be summed to result in

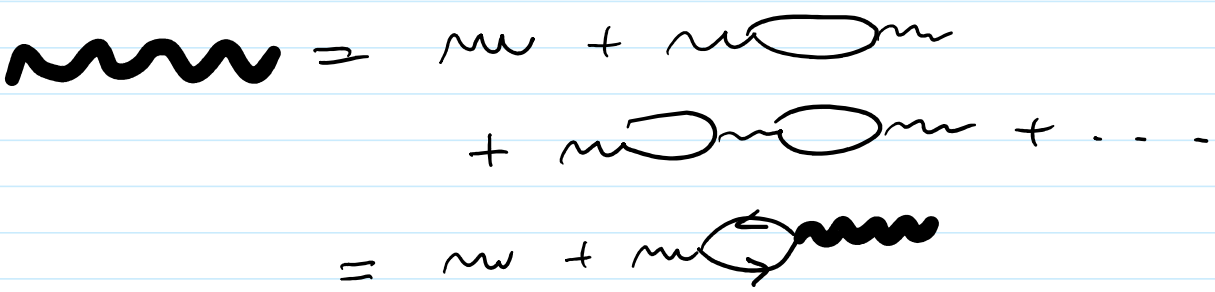


by iterations, it gives me all the above diagrams.

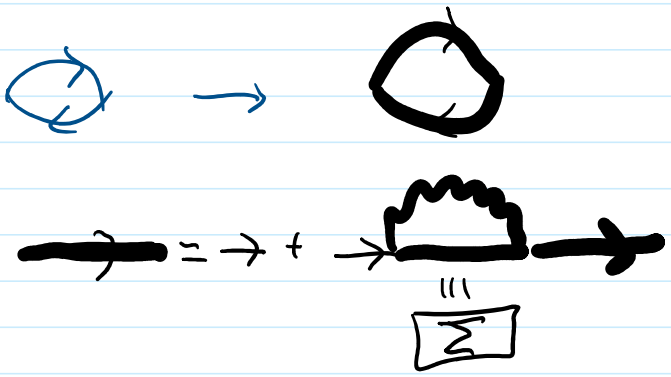
Now we want to add bubbles:



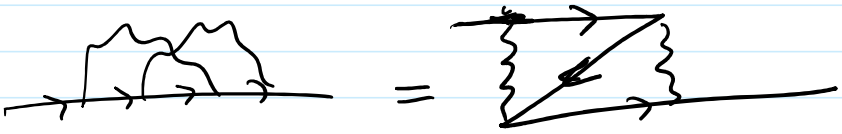
Introduce



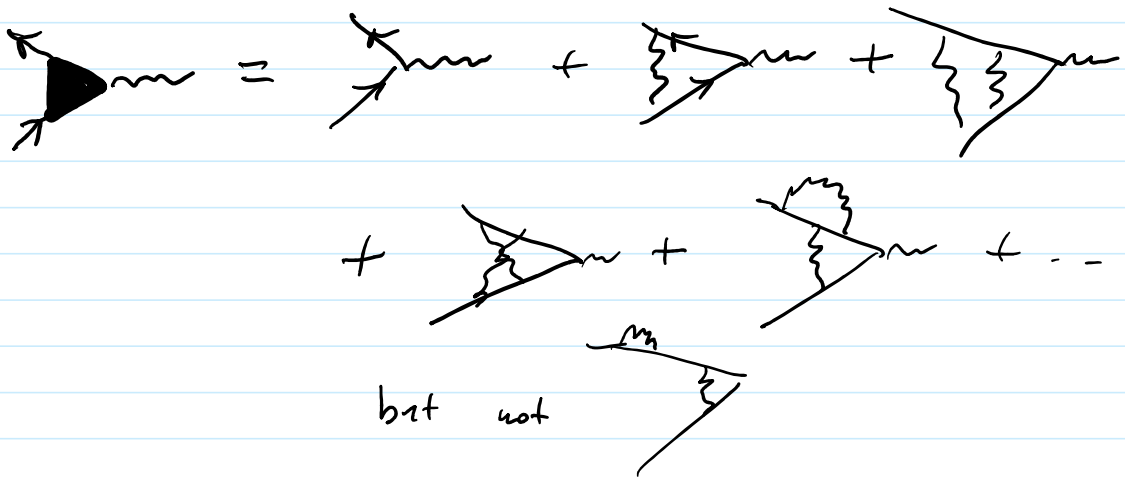
Now we make a bubble from "full" GF



What was left out — diagrams with intersections:

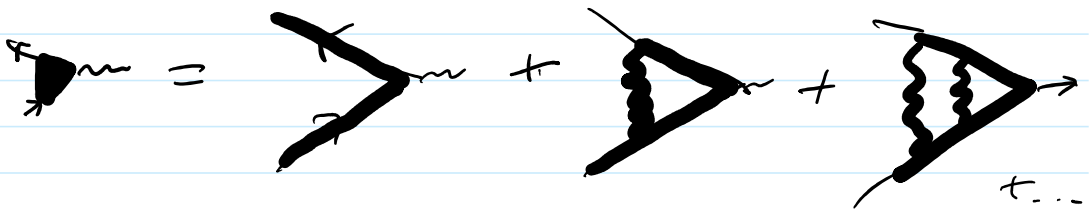
like 

These are called vertex corrections:



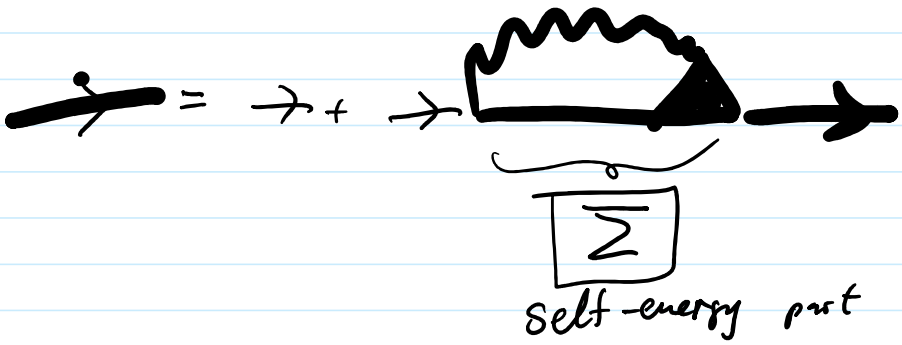
but not

This gives



There's no closed integral eqn for the vertex correction.

Combining all types of corrections, we have an integral eqn with unknown kernel:



$$G(q) = G_0(q) + G_0(q)\,\Sigma(q)\,G(q)$$

Dyson eqn

$$\Rightarrow \quad G_0^{-1} G = 1 + \Sigma G$$

$$\left(G_0^{-1} - \Sigma\right) G = 1 \quad \Rightarrow \quad \underline{G = \left[G_0^{-1} - \Sigma\right]^{-1}}$$

For noninteracting electrons,

$$G_0(x - x'; \omega) = \sum_\alpha \frac{\psi_\alpha^*(\underline{r})\,\psi_\alpha(\underline{r}')}{\varepsilon - (\varepsilon_\alpha - \mu) + i\delta\,\mathrm{sgn}\,\varepsilon}$$

If $\varepsilon_\alpha \to \varepsilon_p = \frac{p^2}{2m}$, then $\psi_\alpha(\underline{r}) = \psi_p(\underline{r}) = e^{\frac{i\underline{p}\underline{r}}{\hbar}}$

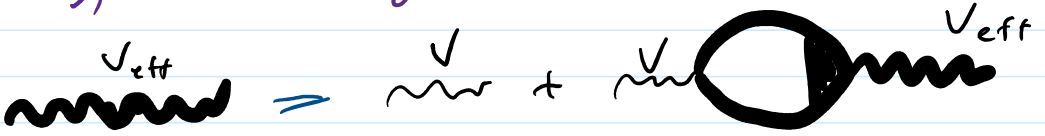
and after the FT we have

$$G_0(q) \equiv G_0(p, \varepsilon) = \frac{1}{\varepsilon - \left(\frac{p^2}{2m} - \mu\right) + i\delta \operatorname{sgn}\varepsilon}$$

$$\Rightarrow \quad G(q) = \frac{1}{\varepsilon - (\varepsilon_q - \mu) - \Sigma}$$

When $\operatorname{Im}\Sigma \neq 0$, we don't need $i\delta \operatorname{sgn}\varepsilon$.

Similarly, we can go to the effective interaction:



Dyson eqn

$\Pi(q)$

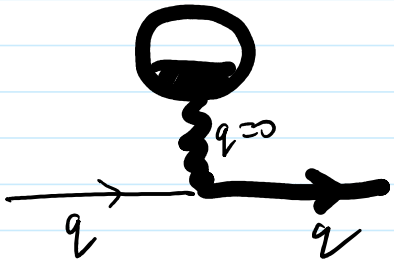
polarisation "operator"

$$V_{eff} = \frac{1}{V^{-1} - \Pi} = \frac{V}{1 - V\Pi}$$

In the lowest order,

$\Pi \to \Pi_0 =$  – contains info about metals

Up to now, we've ignored



$$F(q=0) = \int \tilde{F}(x)\, dx$$

In many-electron systems in metals, the contributions with $q=0$ cancel entirely by

electroneutrality.

Hence, all diagrams with $q=0$ (ie those which can be cut across interaction line) cancel with the contribution of ions (or positively charge substrate).

Now we try to apply general techniques to electrons in metals.

$$V(r) = \frac{\tilde{e}^2}{|r|} = \frac{\tilde{e}^2}{r}, \quad \text{where } \tilde{e}^2 = \begin{cases} e^2, & \text{CGS} \\ \frac{e^2}{\varepsilon}, & \\ \frac{e^2}{4\pi\varepsilon_0\varepsilon} & \text{SI} \end{cases}$$

Let's estimate when such an interaction can be considered as "weak".

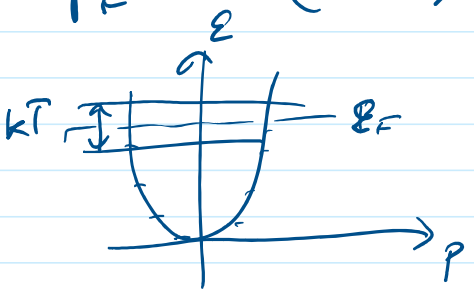
$$\langle \text{Coulomb} \rangle \sim \left\langle \frac{e^2}{r} \right\rangle \sim \frac{e^2}{\lambda_F} \sim e^2 p_F \qquad (\hbar = 1)$$

$$\langle \text{Kinetic energy} \rangle \sim \frac{p_F^2}{m}$$

$$\frac{V_{coul}}{E_{kin}} \sim \frac{e^2 m}{p_F} = \frac{e^2}{v_F} \to \frac{e^2}{\hbar v_F} \sim 1$$

$$\left[\alpha = \frac{e^2}{\hbar c} = \frac{1}{137} \right]$$

$$\frac{e^2}{\hbar v_F} = \underbrace{\frac{m e^2}{\hbar^2}}_{1/a_B} \frac{\hbar}{m v_F} \equiv \frac{\lambda_F}{a_B} \equiv r_s$$



$kT \ll p_F$, we estimate that all electrons have $p \sim p_F$

$1\,eV \sim 10^4\,K$

$\varepsilon_F \sim 1 \div 10\,eV$

The interaction is weak when $r_s = \frac{V_{coul}}{E_{kin}} \ll 1$, ie $\lambda_F \ll a_B$.

The high density is (in principle) possible but we can use $r_s \ll 1$ (high density approximation), and take $r_s \to 1$ at the end of calculations.

The lowest order in the interaction should be formulated for irreducible diagrams, ie those that can't be cut in two across one G_0.

We must calculate Π and/or Σ, and substitute it into G or V_{eff}.

When we choose $\Pi \to \Pi_0 =$  , we have

$V_{eff} =$ 

$$= \frac{V}{1 - V\Pi_0}$$

This approximation is called RPA – random phase ....

It corresponds to summation of all "most divergent" diagrams for the GS energy.