Lecture 6 - Temperature technique. Weakly interacting bosons.

08 March 2023 14:00

Temperature techniques for QFT

At $T=0$, we had

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, e^{iS}$$

$$S = \int dx \, F(\bar{\psi}, \psi) \equiv \int dt \int d\underline{r} \, F(\bar{\psi}, \psi)$$

Now, $iS \to -S$: $Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, e^{-S}$

and $t \to i\tau \longrightarrow$ imaginary time (Wick rotation)

$$S \to \int_0^\beta d\tau \int d\underline{r} \, F(\bar{\psi}, \psi) \; ; \quad \beta = \frac{1}{T}$$

$$S_0 = \int \bar{\psi} \left[i\partial_t - \frac{\nabla^2}{2m} - \mu \right) \psi \, dx \, ;$$

After the Wick rotation,

$$S_0 = \underbrace{\int_0^\beta d\tau \int d\underline{r}}_{\int dx} \; \bar{\psi}(x) \left[\partial_\tau - \frac{\nabla^2}{2m} - \mu \right] \psi(x)$$

$\underline{\psi}(\tau)$ is for $\tau \in [0, \beta]$

Go to Fourier series

$$\psi(\tau) = \frac{1}{\beta} \sum_n \psi(\omega_n) \, e^{i\omega_n \tau}$$

$$\psi(\omega_n) = \int_0^\beta d\tau \; \psi(\tau) \, e^{-i\omega_n \tau}$$

$$\psi(\omega_n) = \int_0^\beta d\tau\, e^{-i\omega_n\tau} \left[\frac{1}{\beta} \sum_m e^{i\omega_m\tau} \psi(\omega_m) \right]$$

$$= \frac{1}{\beta} \sum_m \psi(\omega_m) \int_0^\beta d\tau\, e^{-i(\omega_n-\omega_m)\tau}$$

$$= \frac{1}{\beta} \sum_m \psi(\omega_m) \underbrace{\frac{e^{i(\omega_m-\omega_n)\beta} - 1}{i(\omega_m-\omega_n)}}_{\text{must be } \beta\delta_{mn}}$$

For $m \neq n$, we should have $e^{i(\omega_m-\omega_n)\beta} = 1$

$$\omega_m - \omega_n = 2\pi \times \text{integer} \times T$$

We can choose $\omega_m = 2\pi m T$, $m \in \mathbb{Z}$ $\rightarrow$ true for bosons. (even)

For fermions, $\omega_m = 2\pi T(m+\frac{1}{2}) = \pi T(2m+1)$ (odd)

We'll check that such a prescription works. It will be automatic, if we impose the BCs:

$$\psi(0) = \psi(\beta) \quad \text{(bosons)}$$

or $$\psi(0) = -\psi(\beta) \quad \text{(fermions)}$$

The zeroth appr. GF corresponding to

$$S_0 = \int dx\, \bar{\psi}\left(\partial_\tau - \frac{\nabla^2}{2m} - \mu\right)\psi$$

$$= \frac{1}{\beta L^d} \sum_{nq} \bar{\psi}(\omega_n, q) \Big(-i\omega_n + \underbrace{\overbrace{\frac{q^2}{2m}}^{\varepsilon(q)} - \mu}_{\xi_q \equiv \varepsilon_q - \mu}\Big) \psi(\omega_n, q)$$

$$= -\frac{1}{\beta L^d} \sum_{nq} \bar{\psi}\, G_0^{-1}\, \psi$$

$$\Rightarrow \quad G_0 = \frac{1}{i\omega_n - \xi_q}$$

Check that this reproduces Bose (or Fermi) distribution.

$$\frac{N}{L^d} = \frac{1}{L^d} \int \rho(r)\, d^d r$$

$$= \frac{1}{L^d} \int \bar{\psi}(\underline{r}, \tau)\, \psi(\underline{r}, \tau)\, d^d r \, ;$$

Before (at $T=0$), we had $G = -i \langle \bar{\psi}(x) \psi(x') \rangle$

Now $-i$ disappears, and we have

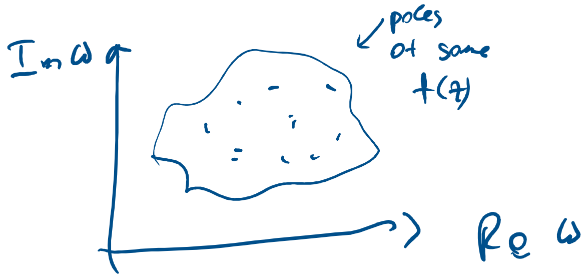
$$\mathcal{G}(\underline{r}, \underline{r}'; \tau - \tau') = \mathcal{G}(\underline{r}, \tau; \underline{r}', \tau') \equiv \mathcal{G}(x, x') = \langle \bar{\psi}(x) \psi(x') \rangle$$

$$\langle \ldots \rangle = \int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, (\ldots)\, e^{-S}$$

Hence, $\rho(\underline{r}) = \frac{1}{L^d} \lim_{\tau \to 0} \int d^d r \; G(\underline{r}, \underline{r}; \tau) =$

$$= \frac{1}{L^d} \frac{1}{\beta} \lim_{\tau \to +0} \sum_n \frac{1}{i\omega_n - \xi_q} e^{i\omega_n \tau}$$

Let's introduce a counting f-n, $\mathcal{D}(z)$, on the complex plane $z = (\mathrm{Re}\,\omega, i\,\mathrm{Im}\,\omega)$



$\mathcal{D}(z) = 0$ in all points of summation

$$\sum_n f(z_n) = \frac{1}{2\pi i} \oint f(z) \frac{\mathcal{D}'(z)}{\mathcal{D}(z)} dz$$

$\mathcal{D}(z_n) = 0$ by definition;

at each zero, $\mathcal{D}(z) = \underset{\overset{\parallel}{0}}{\mathcal{D}(z_n)} + \mathcal{D}'(z=z_n)(z - z_n) + \ldots$

$$\frac{\mathcal{D}'(z)}{\mathcal{D}(z)} \xrightarrow[z \to z_n]{} \frac{1}{z - z_n}$$

Then $\displaystyle \frac{1}{2\pi i} \oint_{\substack{\text{around} \\ z_n}} f(z) \frac{\mathcal{D}'(z)}{\mathcal{D}(z)} dz = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_n} = \underline{f(z_n)}$

$$\sum_n f(z_n) = \frac{1}{2\pi i} \oint f(z) \frac{\mathcal{D}'(z)}{\mathcal{D}(z)} dz$$

includes all zeros

To calculate $\displaystyle \frac{1}{\beta} \sum_n \frac{e^{i\omega_n \tau}}{i\omega_n - \xi_q}$ for bosons,

we choose $\mathcal{D}(z) = e^{\beta z} - 1$

when $\beta z_n = 2\pi i n$, $\mathcal{D}(z_n) = 0$

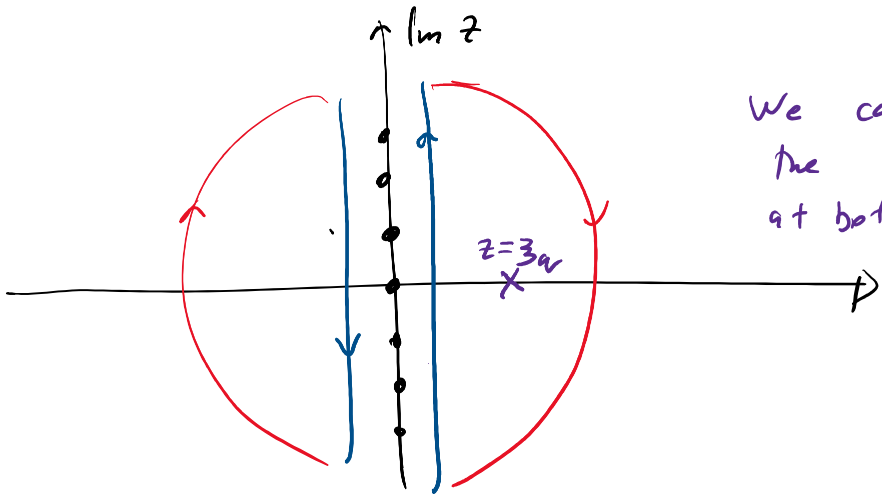
[For fermions, $\mathcal{D}(z) = e^{\beta z} + 1$]

$$\mathcal{D}'(z) = \beta e^{\beta z} \Rightarrow \mathcal{D}'(z_n) = \beta e^{\beta z_n} = \beta$$

$$\therefore \quad \frac{\mathcal{D}'(z)}{\mathcal{D}(z)} = \frac{\beta}{e^{\beta z} - 1}$$

The sum, $\displaystyle \frac{1}{\beta} \sum_n \frac{e^{i\omega_n \tau}}{i\omega_n - \xi_q} = \frac{1}{2\pi i} \oint_C \frac{e^{z\tau}}{z - \xi_q} \frac{1}{e^{\beta z} - 1} dz$

The contour should include all z_n



We can close the contour at both $\mathrm{Re}\, z \to \pm\infty$

For $\mathrm{Re}\, z \to +\infty$,

$$\frac{e^{z\tau}}{z - \xi} \frac{1}{e^{\beta z} - 1} \longrightarrow \frac{e^{z(\tau - \beta)}}{z} \to 0 \text{ as } \tau < \beta$$

For $\mathrm{Re}\, z \to -\infty$, $e^{\beta z} - 1 \to -1$

so the integrand $\dfrac{e^{z\tau}}{z}$

in the lim $\tau \to +0$, $e^{z\tau} \to 0$ for $\mathrm{Re}\, z \ll 1$.

The single pole at $z = \xi_q$ gives

$$\lim_{\tau \to +0} \frac{1}{\beta} \sum \frac{e^{i\omega_n \tau}}{i\omega_n - \xi_q} = \frac{1}{2\pi i} \lim_{\tau \to 0} \oint \frac{e^{z\tau}}{z - \xi_q} \frac{1}{e^{\beta z} - 1}$$

$$= \lim_{\tau \to 0} \frac{e^{\xi_q \tau}}{e^{\xi_q \beta} - 1} = \frac{1}{e^{\beta(\varepsilon_q - \mu)} - 1}$$

Bose distribution.

Similar considerations work for fermions.

As usual, when $S_0 \to S_0 + S_{int}$,

we have a similar expression for $\mathcal{G}(x) \to$ the full GF, where our typical

$$S_{int} = \frac{1}{2} \int \bar{\psi}(x) \bar{\psi}(x') U(x-x') \psi(x') \psi(x) \, dx \, dx'$$

$$V(x-x') = V(\underline{r} - \underline{r}') \delta(\tau - \tau').$$

$$= \frac{1}{2} \int \rho(x) V(x-x') \rho(x') \, dx \, dx'$$

$$\rho(x) = \bar{\psi}(x) \psi(x)!$$

The simplest $\rho - \rho$ interaction is the contact one:

$$V(\underline{r} - \underline{r}') = g \, \delta(\underline{r} - \underline{r}')$$

Then, $S_{int} = \frac{g}{2} \int \rho^2(\underline{r}, \tau) \, dx$

$$= \frac{g}{2} \int \bar{\psi}(x)\psi(x)\,\bar{\psi}(x)\psi(x)\,dx$$

Weakly interacting bosons with $S = S_0 + S_{int}$ and small g.

For $g=0$, we have an ideal Bose gas, which at $T < T_0 \sim \frac{\hbar^2}{m a^2}$ (a is an interparticle distance), condences to the GS with zero energy.

In the condensed phase, $\mu = 0$ for the ideal Bose gas.

Assume that in weakly interacting Bose gas the condensed phase has WF $\underset{\text{condensed}}{\psi(0)} + \underset{\text{excitations}}{\psi_1(x)}$

$\psi(0)$ corresponds to homogeneous Bose condensate

$\psi_1(x)$ "small" above-condensate excitations

$$S = S_0 + S_{int};$$

$$S_0 = \int \bar{\psi}\left(\partial_\tau - \frac{\nabla^2}{2m} - \mu\right)\psi\,dx$$

$$S_0 + S_{int} = \int \bar{\psi}_0(-\mu)\psi_0\,dx + \frac{g}{2}\int (\bar{\psi}\psi)^2\,dx$$

$$\rightarrow \int dx \left[\underset{\substack{\downarrow \\ \bar{\psi}_0 \psi_0 \text{ — condensate density}}}{-\mu \rho_0} \right] + \frac{g}{2} \int \rho_0^2 \, dx$$

Density of the action

$$\mathcal{L}_0 = -\mu \rho_0 + \frac{g}{2} \rho_0^2$$

The minimum of the action ("classical" trajectory)

$$\frac{\partial \mathcal{L}_0}{\partial \rho_0} = 0 \Rightarrow -\mu_0 + g \rho_0 = 0$$

$$\rho_0 = \frac{\mu_0}{g} > 0 \longrightarrow \mu_0 \text{ is positive}$$

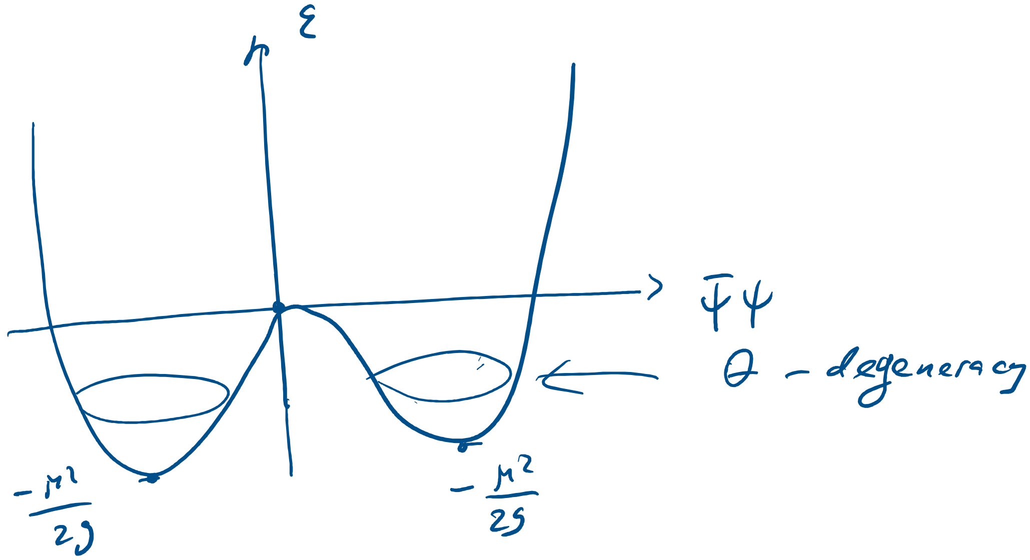
The condensation energy,

$$S_{min} = -\frac{g}{2} \rho_0^2 = -\frac{\mu^2}{2g} < 0$$

Now we need to include ψ_1 to take account of quantum fluctuations.

ψ is a complex field; $\psi = |\psi| e^{i\theta}$

In the min, only $|\psi| = \sqrt{\bar{\psi}_0 \psi_0}$ is fixed, while θ is free.



The θ-modes don't change energy; this is an example of Goldstone's modes.

Let's write

$$\left.\begin{aligned} \psi_0 &= \sqrt{\rho_0}\, e^{i\theta} \\ \bar\Psi_0 &= \sqrt{\rho_0}\, e^{-i\theta} \end{aligned}\right\} \text{ the mean-field result with no restrictions on } \theta.$$

In general ρ is also fluctuating,

$$\psi = \sqrt{\rho_0 + \rho_1}\, e^{i\theta}$$

ρ_1: over-condensate density excitations

The full action — MF + ρ-fluctuations

$$S = \tilde S_0 + S_{int} = \int \bar\Psi(x)\left(\partial_\tau - \frac{\nabla^2}{2m} - \mu\right)\psi \, d\tau$$

ψ: not ψ_0

$$+\frac{g}{2}\int \rho^2\, dx$$

$$\partial_\tau \psi = \partial_\tau\left(\sqrt{\rho_0+\rho_1}\, e^{i\theta}\right) = \frac{\partial_\tau \rho_1}{2\rho_0} e^{i\theta} + i\sqrt{\rho}\, e^{i\theta}\partial_\tau\theta$$

$\uparrow$ neglect ρ_1;

Substitute to $\tilde{S}_0$:

this term vanishes as a full derivative

$$\tilde{S}_0 = \int \left(\frac{\partial_\tau \rho_1}{2\rho_0} + i\sqrt{\rho}\,\partial_\tau\theta\right) dx + S_{cond}$$

$+$ similar contribution from $\frac{\nabla^2}{2m}$.

$$\sqrt{\rho} = \sqrt{\rho_0+\rho_1} = \sqrt{\rho_0}\sqrt{1+\frac{\rho_1}{\rho_0}} = \frac{1}{2}\frac{\rho_1}{\sqrt{\rho_0}}$$

Substitute also S_{int} in terms of ρ_0 & ρ_1

$$\Rightarrow S = S_{cond} + \int dx\left[i\rho_1\partial_\tau\theta + \frac{1}{2m}\left(\frac{(\nabla\rho_1)^2}{4\rho_0} + \rho_0(\nabla\theta)^2\right)\right]$$

$$+\int dx\left(\frac{1}{2}g\rho_1^2 + g\rho_0\rho_1\right)$$

The two fluctuating fields are $\rho_1(x)$ and $\theta(x)$

The action is quadratic in ρ_1 & θ.

We can calculate exactly Z for any quadratic field.

To begin with, we can neglect $\frac{(\nabla\rho_1)^2}{\rho_0}$

as ρ_1 is both slow & small.

Let's integrate out ρ_1 (always possible for a quadratic action), and find the θ-only action.

$$S_{fe}(\rho_1) = \frac{g}{2}\rho_1^2 + i\rho_1 \partial_\tau \theta$$

$$= \underline{\frac{g}{2}\left[\rho_1^2 - 2i\rho_1 \frac{\partial_\tau \theta}{g} - \left(\frac{\partial_\tau \theta}{g}\right)^2\right]}$$

$$+ \frac{1}{2g}\left(\partial_\tau \theta\right)^2$$

$\underline{\quad\cdots\quad}$ is the only ρ_1-dependent part, and it's quadratic.

We shift $\rho_1 \to \rho_1 - i\frac{\partial_\tau \theta}{g}$ to get the full square $\frac{g}{2}\rho_1^2$ to be cancelled with $\frac{1}{Z}$ in any GF.

After making this integration, we arrive at the effective action

$$S_{eff}[\theta] = \frac{1}{2}\int dx \left[\frac{\rho_0}{m}(\nabla\theta)^2 + \frac{1}{g}(\partial_\tau \theta)^2\right]$$

$$= \frac{1}{2}\int d\underline{q}\, \frac{1}{\beta}\sum_n \left[\frac{\rho_0 q^2}{m} + \frac{\omega_n^2}{g}\right]\theta_q \theta_{q}$$

as $\bar{\theta}_q = \theta_{-q}$.

The appropriate GF is

$$g(\omega_n) = \frac{g}{\omega_n^2 + \rho_0 q^2 \frac{g}{m}}$$

Can make analytic continuation to real frequencies: $t \to i\tau$ (before)

$\omega_n \to i\omega$ (now)

(inverse Wick rotation)

$$G(\omega) = \frac{g}{\omega^2 - \rho_0 q^2 \frac{g}{m}}$$

The poles describe over-condensate excitations:

$$\omega = \pm c q, \quad c = \sqrt{\frac{\rho_0 g}{m}}$$

Excitations are linear in q.

Sound-like excitations $\equiv$ superfluidity

Formally, calculate the current:

$$\underline{j} = -\frac{i}{2m}\left(\bar{\psi}\nabla\psi - \nabla\bar{\psi}\,\psi\right)$$

$$\psi = \rho_0 e^{i\theta} \; ; \; \bar{\psi} = \rho_0 e^{-i\theta}$$

Substitute to get

$$\underline{j} = \frac{\rho_0}{m} \nabla\theta$$

As θ is freely rotating — no decay in current $\equiv$ superfluidity.