- Lecture 7 - QFT for superconductivity 1. Coulomb repulsion against electron-phonon attraction. 7
  - 15 March 2023

Last time: the temperature technique

$$\int dt \to \int_0^\beta d\tau; \qquad t \to i\tau \quad \text{(Wick rotation)}$$

$$\varepsilon \to i\varepsilon_n$$

Matsubara frequencies $\varepsilon_n = \begin{cases} 2\pi n T & \text{(bosons)} \\ \pi(2n+1)T & \text{(fermions)} \end{cases}$

$$\underset{T=0 \text{ action}}{iS} \to \underset{\text{temperature action with the above rotation.}}{-S}$$

The noninteracting action

$$S_0 = \int \bar{\psi} G_0 \psi \, dx; \qquad \int dx \equiv \int_0^\beta d\tau \int d\underline{r}$$

$$G_0 = \frac{1}{i\omega_n - \xi_{\underline{k}}}, \qquad \xi_{\underline{k}} = \varepsilon_{\underline{k}} - \mu \to \frac{k^2}{2m} - \varepsilon_F = \frac{(k-k_F)(k+k_F)}{2m} \approx$$

$$\approx \begin{cases} v_F(|\underline{k}| - k_F) \\ -v_F(|\underline{k}| + k_F) \end{cases}$$

$k = |\underline{k}|$;

__Today:__ ① e-ph + Coulomb interactions

② Debye model for phonons + e-ph interaction.

③ e-ph → attraction (at low energies)
Coulomb → repulsion.

When the attraction can win?

Steps: (a) integrating out the phonons
(b) Hubbard – Stratonovich with symmetry breaking
(c) integrating out "fast" degrees of freedom
– RG (renormgroup).

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(1) Phonons – lattice oscillations –
collection of harmonic oscillators;
this creates effective dipoles, and with the appropriate polarisation interacts
the electron density

Interaction $\sim (e \rho_{el}) |\underline{P}|$

$$\underline{P} = e \, \mathrm{div} \, \underline{d}$$

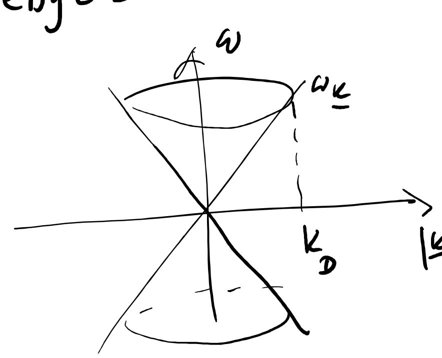
$\underline{d}$: lattice displacement.

$$(H - \omega) \mathcal{D} = \hat{I}$$

$\mathcal{D}$: phonon Green's function

$$H = \sum_{k} \omega_{\underline{k}} (n_{\underline{k}} + \tfrac{1}{2})$$

Debye's model

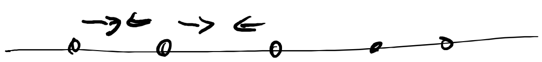


$$\omega_{\underline{k}} = \begin{cases} c|\underline{k}| \equiv ck, & k < k_D \\ 0, & k > k_D \end{cases}$$

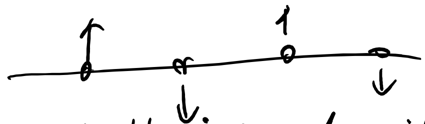
$\omega_D = c k_D$

c speed of sound.

Longitudinal phonons, as $\mathrm{div}\,\underline{d} \neq 0$.



longitudinal
(change local charge density)



transverse $(\mathrm{div}\,\underline{d} = 0)$

don't interact with electrons

As for electrons, $\mathcal{D} = \frac{1}{\omega - \omega_k + i\delta\,\mathrm{sgn}\,\omega} - \frac{1}{\omega + \omega_k + i\delta\,\mathrm{sgn}\,\omega}$

$$= \frac{2\omega_k}{\omega^2 - \omega_k^2 + i\delta}$$

The S_{ph} would be $\int \Phi\, \mathcal{D}^{-1} \Phi\, dx$

Taking $\varphi \sim \frac{\Phi}{\sqrt{2\omega_k}}$, and rescaling

$$\mathcal{D} \to \frac{\omega_k^2}{\omega^2 - \omega_k^2 + i\delta}$$

we have

$$S_{ph} = \frac{1}{2}\int \varphi(x) \underbrace{\mathcal{D}^{-1}(x)}_{\text{FT of the above.}} \varphi(x)\, dx$$

Making the Wick rotation (T - technique),

We have $Z = \int e^{-S_{ph}} \mathcal{D}\varphi$,

S_{ph} is as above but with $\omega \to i\omega_n$ in $\mathcal{D}$.

$$\mathcal{D}(\omega_n, \underline{k}) = -\frac{\omega_k^2}{\omega_n^2 + \omega_k^2} \quad ; \quad \omega_n = 2\pi n T$$

Why to introduce dimensionless φ & $\mathcal{D}$?

E-ph interaction looks simple:

$$S_{e-ph} = g \int dx \, \rho(x) \underbrace{\varphi(x)}_{\text{polarisation magnitude.}} \quad ; \quad \rho(x) = \bar{\psi}(x)\psi(x)$$

Let's start with

$$S = \underbrace{S_0}_{\text{electrons}} + S_{ph} + S_{e-ph}$$

which is quadratic in both φ & ψ.

Integrate out phonons (φ) to get purely electron action:

$$e^{-(S_{ph} + S_{e-ph})} \sim e^{-\int dx \left[\frac{1}{2}\varphi \mathcal{D}^{-1} \varphi - g\varphi\rho\right]}$$

Complete the square:

$$\frac{1}{2\mathcal{D}}\left(\varphi^2 - 2g\varphi\rho\mathcal{D} + g^2\rho^2\mathcal{D}^2\right) - \frac{g^2\rho^2\mathcal{D}}{2}$$

Hence $\varphi \to \varphi - g\rho\mathcal{D}$, and the same shift in calculating GF and $\frac{1}{Z}$ cancel this term and we are left with effective electron action:

$$S_{eff} = \frac{g^2}{2}\int \rho(x)\mathcal{D}(x-x')\rho(x')\,dx\,dx'$$

$$g^2 = \frac{2\pi}{mp_F}\eta \; ; \quad \eta \sim 1 \text{ in metals.}$$

$$\mathcal{D}(\omega_n, \underline{q}) = -\frac{c^2k^2}{\omega_n^2 + c^2k^2} \to -1 \text{ when } T \to 0.$$

$$\mathcal{D}(x-x') \sim -\delta(x-x') \to$$

instantenous local interaction.

$$S_{el-ph} \to -\frac{\lambda}{2}\int \rho^2(x)\,dx \qquad \text{BCS interaction}$$

$$\rho(x) = \bar{\psi}_\sigma(x)\psi_\sigma(x) = \bar{\psi}_\uparrow\psi_\uparrow + \bar{\psi}_\downarrow\psi_\downarrow$$

As the interaction is local, $\psi_\uparrow\psi_\uparrow = \psi_\downarrow\psi_\downarrow = 0$.

In ρ^2 only the singlet term survives

$$S_{el-ph} = -\lambda\int \bar{\psi}_\uparrow\psi_\uparrow\bar{\psi}_\downarrow\psi_\downarrow\,dx$$
$$= -\lambda\int \bar{\psi}_\uparrow\bar{\psi}_\downarrow\psi_\downarrow\psi_\uparrow\,dx$$

Now we need to add Coulomb — screened.

The screening radius (Debye) $r_D^2 = \frac{1}{4\pi\nu e^2} = r\lambda_F a_B$

In metals, $\lambda_F \sim a_B \sim a_0$ — lattice spacing.

e-e interaction $U(x) = U_0\delta(x)$, $U_0 \sim \frac{e^2}{a_0}$

Competition.

$$S_{el-el} = U_0\int dx\,\bar{\psi}_\uparrow\bar{\psi}_\downarrow\psi_\downarrow\psi_\uparrow \qquad \text{repulsion}$$

$$S_{el-ph} \equiv S_{BCS} = -\lambda\int dx\,\bar{\psi}_\uparrow\bar{\psi}_\downarrow\psi_\downarrow\psi_\uparrow \qquad \text{attraction}$$

$S_{BCS} = 0$ for $\varepsilon > \omega_D$

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For $S_{el-el} \neq 0$ we have for $\varepsilon \lesssim \varepsilon_F$
(up to bandwidth).

Let's compare ω_D and ε_F.

$$\omega_D = c k_D \ ; \quad k_D \sim \frac{2\pi}{a_0} \sim \frac{2\pi}{\lambda_F}$$

In metals, electrons & ions are in equilibrium

$$\frac{Mc^2}{2} \sim \frac{m v_F^2}{2} \Rightarrow \frac{c}{v_F} \sim \sqrt{\frac{m}{M}} \ll 1$$

$$\frac{\omega_D}{\varepsilon_F} = \frac{c k_D}{p_F v_F} \approx \frac{c}{v_F} \sim \sqrt{\frac{m}{M}} \ll 1$$

Next task — to derive effective action which valid for energies ε, such $T \lesssim \varepsilon \lesssim \omega_D$.

We want to integrate out all energies $\omega_D \leq \varepsilon \leq \varepsilon_F$.

Next steps for this:

1. Hubbard-Str. with symmetry breaking
2. RG to integrate out high-energy DoF by "slices".

We introduce H-S fields

$$\Delta \sim \psi_\uparrow \psi_\downarrow$$

$$\bar{\ } \quad \bar{\ }$$

$$\Delta \sim \psi_\uparrow \psi_\downarrow$$

$$\bar\Delta \sim \bar\psi_\downarrow \bar\psi_\uparrow$$

Perturbatively,

$$\langle \Delta \rangle = \langle \bar\Delta \rangle = 0$$

in all orders

HS allows for non-perturbative contributions:

We introduce Δ & $\bar\Delta$ as auxilliary fields

such that

$$e^{-S_{el\text{-}el}} = e^{-U_0 \int \bar\psi_\uparrow \bar\psi_\downarrow \psi_\downarrow \psi_\uparrow dx}$$

$$= \frac{1}{Z_\Delta} \int \mathcal{D}\bar\Delta\, \mathcal{D}\Delta\, e^{\underbrace{-\int dx \left[\frac{\bar\Delta \Delta}{U_0} + i\Delta \bar\psi_\uparrow \bar\psi_\downarrow + i\bar\Delta \psi_\downarrow \psi_\uparrow \right]}_{S_\Delta}}$$

Indeed,

$$\frac{1}{U_0}\left[\bar\Delta + i U_0 \bar\psi_\uparrow \bar\psi_\downarrow \right]\left[\Delta + i U_0 \psi_\downarrow \psi_\uparrow \right]$$

$$+ U_0 \bar\psi_\uparrow \bar\psi_\downarrow \psi_\downarrow \psi_\uparrow \equiv \text{as above}$$

If integrate over $\bar\Delta \Delta$, the shift is compensated by $\frac{1}{Z_\Delta}$.

Next task — the $S_0 + S_{coul}$ is quadratic in ψ, so that we can integrate $\psi, \bar\psi$ out when $\varepsilon > \omega_D$

$$S_0 = \int \bar{\psi}_\sigma G_0^{-1} \psi_\sigma \, dx$$

To capture both diagonal & off-diagonal in $\sigma = \uparrow, \downarrow$ terms, introduce

$$\underline{\underline{\psi}}(x) = \begin{pmatrix} \psi_\uparrow \\ \bar{\psi}_\downarrow \end{pmatrix}, \quad \underline{\underline{\bar{\psi}}} = \left(\bar{\psi}_\uparrow \;\; \psi_\downarrow \right)$$

Can check by direct multiplication that

$$S_{eff} = S_0 + S_\Delta =$$

$$= \frac{1}{U_0} \int \bar{\Delta}(q)\, \Delta(q)\, dq \qquad (q \equiv (\omega, \underline{q}))$$

$$+ \int \underline{\underline{\bar{\psi}}}(q)\, \hat{\mathcal{G}}^{-1}\, \underline{\underline{\psi}}(q)\, dq$$

where
$$\hat{\mathcal{G}}^{-1} = \begin{pmatrix} -i\varepsilon_n + \xi_{\underline{q}} & i\Delta \\ i\bar{\Delta} & -i\varepsilon_n - \xi_{\underline{q}} \end{pmatrix}$$

Now we integrate over $\bar{\underline{\underline{\psi}}}, \underline{\underline{\psi}}$ in the infinitesimal energy slice from $U(\Lambda)$ to $U(\Lambda - d\Lambda)$

Λ and $\Lambda - d\Lambda$ are energies between which we integrate; $U(\Lambda_0) = U_0$.

The quadratic action gives $\int \bar{\underline{\underline{\psi}}}\, \mathcal{G}^{-1}\, \underline{\underline{\psi}}\, dq \quad + \ln \det \mathcal{G}^{-1} = e^{-S_{eff}}$

The quadratic action gives

$$\int \mathcal{D}\bar{\underline{\psi}}\,\mathcal{D}\underline{\psi}\; e^{-\int \bar{\underline{\psi}}\, \mathcal{G}^{-1} \underline{\psi}\, dq} = e^{+\ln\det \mathcal{G}} \equiv e^{-S_{eff}}$$

$$S_{eff} = -\ln\det \mathcal{G}^{-1} = -\mathrm{Tr}\ln \mathcal{G}^{-1} =$$

$$= -\int dq\; \mathrm{tr}\ln \mathcal{G}^{-1} = -\int dq\; \ln\det \mathcal{G}^{-1}$$

$$= -\int dq\; \ln\Big[\underbrace{-\varepsilon_n^2 - \xi_k^2}_{-G_0^{-1}} + \bar{\Delta}\Delta\Big]$$

$$= -\int dq\; \ln\big[1 - G_0 \bar{\Delta}\Delta\big]$$

$$\approx \bar{\Delta}\Delta \int dq\; G_0$$

$$= \bar{\Delta}\Delta \frac{1}{L^d}\sum_q \underbrace{\frac{1}{\beta}\sum_{\omega_n} \frac{1}{\omega_n^2 + \xi_{\underline{k}}^2}}$$

$$\frac{1}{\beta}\sum_{\omega_n} \to \int \frac{d\omega}{2\pi} \quad \text{when } T \to 0$$

(exact integral includes $\tanh \frac{\varepsilon}{2T}$).

$$\int \frac{d\omega}{\omega^2 + \xi_k^2}\frac{1}{2\pi} = \frac{1}{2}\frac{1}{|\xi_k|}$$

We are left with

$$\bar{\Delta}\Delta \frac{1}{L^d}\sum_k \frac{1}{2|\xi_k|} = \frac{\bar{\Delta}\Delta}{2}\int_{\Lambda_{dk}}^{\Lambda} \frac{d^dk}{|\xi_{\underline{k}}|}$$

we ...

$$\bar\Delta\Delta \frac{1}{L^d}\sum_k \frac{1}{2|\xi_k|} = \frac{\bar\Delta\Delta}{2}\int_{\Lambda-d\Lambda} \frac{\ldots}{|\xi_k|}$$

$$= \frac{\bar\Delta\Delta}{2}\, 2\nu \int_{\Lambda-d\Lambda}^{\Lambda}\frac{d\xi}{|\xi|} = \bar\Delta\Delta\cdot\nu \ln\frac{\Lambda}{\Lambda-d\Lambda}$$

$$= \bar\Delta\Delta\, \nu\, \frac{d\Lambda}{\Lambda}$$

The expansion of ln is justified as the result is $\ll 1$.

Can't integrate from ω_D to ξ_F as $\ln\frac{\xi_F}{\omega_D} >> 1$ and the expansion of ln det isn't possible.

After the integration, the quadratic action becomes $\frac{1}{\tilde U}\int \bar\Delta\Delta\, dx$

where $\frac{1}{\tilde U} = \frac{1}{U(\Lambda-d\Lambda)} - \nu\frac{d\Lambda}{\Lambda}$

Hence, we have the shift

$$\frac{1}{U(\Lambda)} = \frac{1}{U(\Lambda-d\Lambda)} - \nu\frac{d\Lambda}{\Lambda}$$

$$\nu\frac{d\Lambda}{\Lambda} = \frac{1}{U(\Lambda-d\Lambda)} - \frac{1}{U(\Lambda)} = \frac{U(\Lambda)-U(\Lambda-d\Lambda)}{U^2(\Lambda)}$$

$$= \frac{dU}{U^2}$$

Hence, dif. eq.

$$\nu\frac{d\Lambda}{\Lambda} = \frac{dU}{U^2}$$

... limits $\xi_F \to \omega_D$.

$\delta \quad \Lambda \quad U^2$

We integrate with the limits $\varepsilon_F \to \omega_D$:

$$-\frac{1}{U} = \nu \ln \Lambda \Big|_{\omega_D}^{\varepsilon_F}$$

$$\nu U(\omega_D) = \frac{\nu U_0}{1 + \nu U_0 \ln \frac{\varepsilon_F}{\omega_D}}$$

In the limit $\ln \frac{\varepsilon_F}{\omega_D} \gg 1$,

$$\nu U(\omega_D) = \frac{1}{\ln \frac{\varepsilon_F}{\omega_D}} \ll 1.$$

The overall action for $\varepsilon < \omega_D$ becomes

$$S = S_0 - \mu \int \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \, dx$$

where $\mu = \lambda_{BCS} - \frac{1}{\nu \ln \frac{\varepsilon_F}{\omega_D}}$ $\to$ if it's positive, it's BCS with renormalised attraction.

$\uparrow$

MacMillan const

If $\mu < 0$ $\to$ it's repulsion and no BCS action.