Lecture 8 - Cooper instability. Saddle-point action. Ginzburg-Landau functional.

22 March 2023 14:00

Last time:

$$S = S_0 + S_{coul} + S_{BCS}$$

Used H-S with $\Delta \sim \psi_\downarrow \psi_\uparrow$ & $\bar{\Delta} \sim \bar{\psi}_\uparrow \bar{\psi}_\downarrow$ to get the action,

$$S = \frac{1}{U} \int \bar{\Delta}(q) \Delta(q)\, dq + \int \hat{\bar{\psi}}\, \hat{g}^{-1} \hat{\psi}\, dx$$

$$\hat{\psi} = \begin{pmatrix} \psi_\uparrow \\ \bar{\psi}_\downarrow \end{pmatrix} \qquad \hat{\bar{\psi}} = \left(\bar{\psi}_\downarrow \;\; \psi_\uparrow \right)$$

$$\hat{g} = \begin{pmatrix} -i\varepsilon_n + \xi_q & i\Delta \\ i\bar{\Delta} & -i\varepsilon_n - \xi_q \end{pmatrix}$$

After slied integration, we've got the effective action which has a BCS form with

$$\lambda_{eff} = \lambda_{BCS} - \frac{1}{\nu \ln \frac{\varepsilon_F}{\omega_D}} \rightarrow \lambda > 0 \text{ for } \varepsilon \lesssim \omega_D$$

describes effective attraction.

$$S_{BCS} = S_0 - \underset{\substack{\downarrow \\ \text{positive}}}{\lambda} \int \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \, dx$$

In contrast, Coulomb has the same form with $\lambda \to -U_0$

We again do H-S T but with the opposite sign in the quartic term, $U_0 \to -\lambda$, and with energies $< \omega_D$.

Decoupling with Δ & $\bar{\Delta}$ behaving like before

$$e^{\lambda \int \bar{\psi}_\downarrow \bar{\psi}_\uparrow \psi_\uparrow \psi_\downarrow \, dx}$$

$$= \int \mathcal{D}\bar{\Delta}\mathcal{D}\Delta \; e^{-\int dx \left[\frac{1}{\lambda} |\Delta|^2 - \bar{\Delta} \psi_\downarrow \psi_\uparrow - \Delta \bar{\psi}_\uparrow \bar{\psi}_\downarrow \right]}$$

no imaginary i

The effective action quadratic in ψ:

$$S_{eff}(\Delta, \bar{\Delta}; \bar{\psi}, \psi) = \frac{1}{\lambda} \int dq \left[\bar{\Delta}\Delta + \hat{\bar{\psi}} \mathcal{G}^{-1} \hat{\psi} \right]$$

$dq = d\underline{p}\, d\omega$, and $|\omega| < \omega_D$

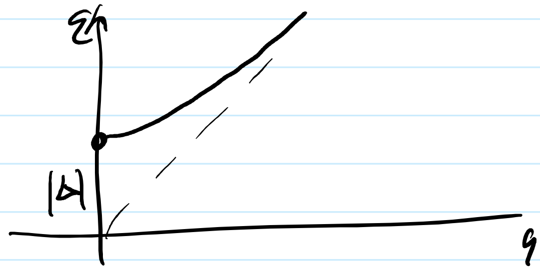
$$\mathcal{G}^{-1} = \begin{bmatrix} -i\varepsilon_n + \xi_{\underline{q}} & -\Delta \\ -\bar{\Delta} & -i\varepsilon_n - \xi_{\underline{q}} \end{bmatrix}$$

$$\mathcal{G} = \frac{1}{\varepsilon_n^2 + \xi_{\underline{q}}^2 + |\Delta|^2} \begin{bmatrix} i\varepsilon_n + \xi_{\underline{q}} & -\Delta \\ -\bar{\Delta} & i\varepsilon_n - \xi_{\underline{q}} \end{bmatrix}$$

The analytic continuation to real frequency, $i\varepsilon_n \to \varepsilon$

$$\Rightarrow \quad G = \frac{1}{\varepsilon^2 - (\xi_q^2 + |\Delta|^2)} \begin{bmatrix} \varepsilon + \xi_{\underline{q}} & -\Delta \\ -\bar{\Delta} & \varepsilon - \xi_{\underline{q}} \end{bmatrix}$$

The poles of G give excitations; if we assume that $|\Delta|^2$ is just a number — not field:



a gap in the Spectrum

As we have done for the Coulomb, we integrate out $\bar{\Psi}\psi$ by slices;

Then, we have the RG eqn as before, but with the opposite sign, and integration between T & ω_D

$$\nu\lambda[T] = \frac{\nu\lambda}{1-\nu\lambda \ln\frac{\omega_D}{T}}$$ Cooper's instability

$$\left[\text{Compare with } \nu U(\omega_D) = \frac{\nu U_0}{1+\nu U_0 \ln\frac{\varepsilon_F}{\omega_D}} \right]$$

Obviously, we have divergence at $\frac{\omega_D}{T} = \exp\frac{1}{\nu\lambda}$

The expansion to the lowest power of $\Pi_r \ln$, we assume that the small parameter was there.

Now we need to sum over most divergent terms in all orders. (Gor'kov)

BCS → used the variational approach.

From the QFT viewpoint, we simply started the expansion from the "wrong" initial point: keeping only S_0 in the integrals over action.

QFT: try to find the minimal action, and then expand in the fluctuations around the minimum.

Minimisation:

$$\frac{\delta S}{\delta \Delta} = 0 \qquad \text{where we assume } \Delta = \underset{\uparrow \atop \text{Optimum}}{\Delta_0} + \underbrace{\tilde{\Delta}(\omega, q)}_{\text{fluctuations}}$$

Here $S = \int dq \left[\frac{\bar{\Delta}\Delta}{\lambda} + \hat{\bar{\psi}} \hat{g}^{-1} \hat{\psi} \right] = S_0 + S_{\psi\Delta}$

Variational derivatives: $\frac{\delta \Delta(x)}{\delta \Delta(x')} = \delta(x - x')$; $\frac{\delta \Delta}{\delta \bar{\Delta}} = 0$

or similar in q-representation.

$$\frac{\delta S_0}{\delta \Delta} = \frac{\delta}{\delta \Delta} \left[\frac{1}{\lambda} \int dx\, \bar{\Delta} \Delta \right] = \frac{\bar{\Delta}}{\lambda}$$

The $S_{\psi\Delta}$ can be integrated over $\bar{\psi}\psi$

giving $\det \hat{g}^{-1} = e^{\ln \det \hat{g}^{-1}} = e^{\mathrm{Tr} \ln \hat{g}^{-1}} = e^{-S_{int}}$

$$\frac{\delta S_{int}}{\delta \Delta} = -\frac{\delta}{\delta \Delta} \mathrm{Tr} \ln \hat{g}^{-1} = -\mathrm{Tr} \frac{\delta}{\delta \Delta} \ln \hat{g}^{-1}$$

$$= -\mathrm{Tr}\, \hat{g} \frac{\delta}{\delta \Delta} \hat{g}^{-1} = + \mathrm{Tr}\, \hat{g} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \mathrm{Tr}\, \hat{g}_{21}$$

Hence, combining the two:

$$\frac{\bar{\Delta}}{\lambda} = -\mathrm{Tr}\, \hat{g}_{21} = \mathrm{Tr} \frac{\bar{\Delta}}{\varepsilon_n^2 + \xi_{\underline{q}}^2 + |\Delta|^2}$$

$$= \int dq \frac{\bar{\Delta}}{\varepsilon_n^2 + \xi_q^2 + |\Delta|^2} \qquad \text{Here } \Delta \to \Delta_0$$

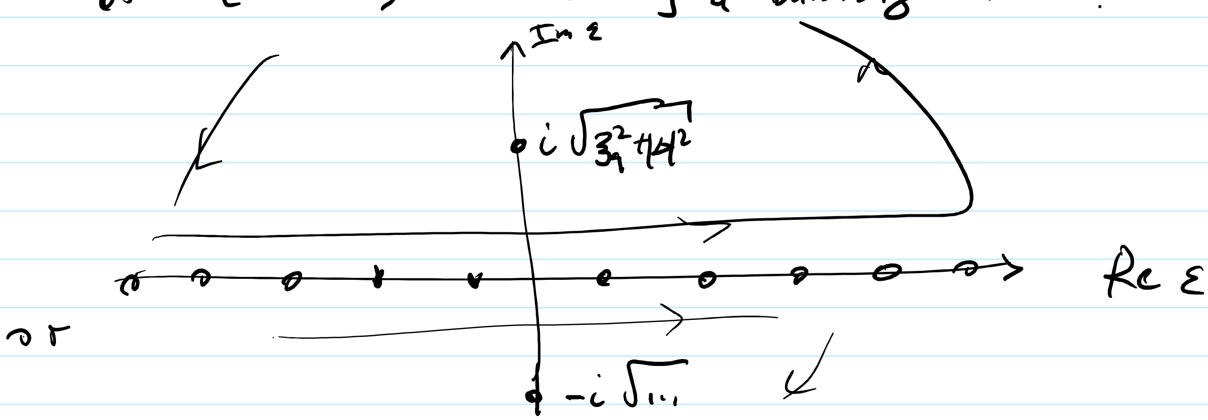
$\Delta = \Delta_0 + \tilde{\Delta} \to$ fluctuations.

Thus we arrive at the self-consistency eq.:

$$\frac{1}{\lambda} = \frac{1}{\beta} \frac{1}{L^d} \sum_{\varepsilon_n \underline{q}} \frac{1}{\varepsilon_n^2 + \xi_q^2 + |\Delta_0|^2}$$

Let's sum over $\varepsilon_n = 2\pi T (n + \frac{1}{2})$

The sum goes to the integral on the complex plane of ε by introducing a counting f-n:



$$T \sum f(\varepsilon_n) = \frac{1}{2}\frac{1}{2\pi i} \oint f(z)\, D(z)$$

$\mathcal{D}(z)$ has poles at all $\varepsilon_n = \pi T(2n+1)$

and $\frac{\mathcal{D}'(z_n)}{\mathcal{D}(z_n)} = 1$

$$D(z) = \tan\frac{z}{2T};$$ when $z \to \pm i\infty$, $\tan\frac{z}{2T} \to \tanh(\infty) \cdot 1$

$$T\sum_{\varepsilon_n} \frac{1}{\varepsilon_n^2 + \xi^2 + |\Delta_0|^2} = \frac{1}{2\pi i} \oint \frac{1}{z^2 + (\xi_q^2 + \Delta_0^2)} \tan\frac{z}{2T}$$

$$= \frac{1}{2\pi i}\, 2\pi i\, \frac{1}{i2\sqrt{\xi^2+\Delta_0^2}} \tan\frac{i\sqrt{\dots}}{2T}$$

$$= \frac{1}{2\sqrt{}} \tanh\frac{\sqrt{}}{2T}$$

Substitute into self-cons eqn:

$$\frac{1}{\lambda} = \frac{1}{2} \int d\bar{q}\, \frac{\tanh\frac{\sqrt{}}{2T}}{\sqrt{\xi^2+\Delta_0^2}} =$$

$$= \frac{\nu}{2} \int_{-\omega_D}^{\omega_D} d\xi\, \frac{\tanh\frac{\sqrt{\xi^2+\Delta_0^2}}{2T}}{\sqrt{\xi^2+\Delta_0^2}}$$

When $T \to 0$, $\tanh \to 1$, so that

$$\frac{1}{\lambda} = \nu \int_0^{\omega_D} \frac{d\xi}{\sqrt{\xi^2+\Delta_0^2}} = \nu \sinh^{-1}\frac{\xi}{\Delta}\Big|_0^{\omega_D}$$

$$\Rightarrow \sinh\frac{1}{\nu\lambda} = \frac{\omega_D}{|\Delta_0|}$$

$$\Rightarrow \quad |\Delta|_0 = \frac{\omega_D}{\sinh \frac{1}{\nu\lambda}} \cong \omega_D e^{-\frac{1}{\nu\lambda}} \ll \omega_D .$$

If $T \gg \omega_D$,

$$\frac{1}{\nu\lambda} = \int_{-\omega_D}^{\omega_D} \frac{1}{2T} d\xi = \frac{\omega_D}{T} \longrightarrow \text{wrong}$$

$\rightarrow$ no solution for large T.

Hence, there exists T_c where Δ_0 starts to deviate from 0.

$$\Delta_0(T_c) = 0 \quad \text{and} \quad \Delta_0(T < T_c) \neq 0 .$$

$$\underline{\frac{1}{\nu\lambda}} = \int_0^{\omega_D} d\xi \, \frac{\tanh \frac{\xi}{2T_c}}{\xi} = \int_0^{\frac{\omega_D}{2T_c}} du \, \frac{\tanh u}{u} \qquad \frac{\xi}{2T} = u$$

$$\sim \int_1^{\frac{\omega_D}{2T}} du \, \frac{\tanh u}{u} \sim \int_1^{\frac{\omega_D}{2T_c}} \frac{du}{u} = \ln \underline{\frac{\omega_D}{2T_c}}$$

$$\Rightarrow \quad T_c = \frac{\omega_D}{2} e^{-\frac{1}{\nu\lambda}} \sim |\Delta_0|$$

The full effective functional should describe fluctuations, $\Delta = \Delta_0 + \Delta_{fl}$

Let's write $\mathcal{G}^{-1}(x)$ as

$$\hat{\mathcal{G}}^{-1} = \underbrace{\begin{bmatrix} -\partial_\tau + \hat{\xi} & 0 \\ 0 & -\partial_\tau + \hat{\xi} \end{bmatrix}}_{\mathcal{G}_0^{-1}} - \underbrace{\begin{bmatrix} 0 & \Delta \\ \bar{\Delta} & 0 \end{bmatrix}}_{\hat{\Delta}}$$

We expand this in small Δ_{fe}:

$$S - S_0 = -\mathrm{Tr} \ln \hat{\mathcal{G}}^{-1} = -\mathrm{Tr} \ln \mathcal{G}_0^{-1} \left[1 - \mathcal{G}_0 \hat{\Delta}\right]$$

$$= \underbrace{-\mathrm{Tr} \ln \mathcal{G}_0^{-1}}_{\text{irrelevant}} - \mathrm{Tr} \ln \left(1 - \mathcal{G}_0 \hat{\Delta}\right)$$

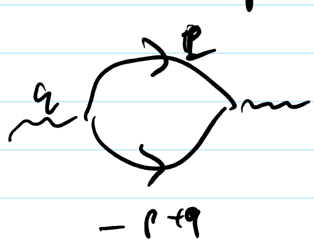
$$= +\mathrm{Tr} \sum_n \frac{1}{n} \left(\hat{\mathcal{G}}_0 \hat{\Delta}\right)^n$$

In the lowest order,

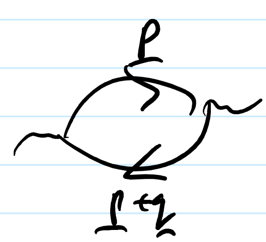
$\mathrm{Tr} \left(\mathcal{G}_0 |\Delta|\right)^2 \rightarrow$ the lowest order in which this contributes

$$\Rightarrow \int dq \; \bar{\Delta}_q \Delta_q \Pi_+(q)$$

$$\Pi_+(q) = \sum_p G_0(p) G_0(-p+q)$$



Cooperon
propagation of e-e pair

$\Pi_- =$ 

polarization operator
(e-h pair)

Singular when $\underline{q} \to 0$
the "sum" momentum

Singular when
$q \to 0$
transferred
momentum

We have action $\sim \sum \bar{\Delta}_q \Gamma_q \Delta_q$

where $\Gamma_q = \left(\frac{1}{\lambda} - \Pi_f\right)$

When we expand the result of calculations of Π_f in small $\underline{q}$ and go into the x-representation,

we end up with

$$S = \int dx \left[\frac{a T_0}{2} |\Delta|^2 + \frac{c}{2} |\partial \Delta|^2 \right]$$

where $a = a_0 (T - T_c)$

We can add e-m field by the minimal substitution,

$$\partial \to \partial - i e^* \underline{A} \quad \text{(Ginzburg-Landau).}$$

$e^* = 2e$ — the charge of the Cooper pair.

The next term of the expansion gives $\sim |\Delta|^4$, that stabilizes the action.