Lecture 9 - QFT for phase transitions in spin systems

14:00

Classical (and later quantum) FT of phase transitions.

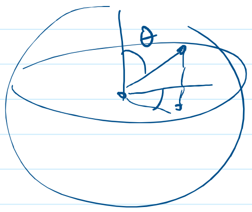
QFT representation for spin system:

$$|\sigma\rangle = a|\uparrow\rangle + b|\downarrow\rangle$$

In 3D,

$$|\sigma\rangle = e^{ib}\left(e^{-\frac{i\varphi}{2}}\cos\frac{\theta}{2}|\uparrow\rangle + e^{\frac{i\varphi}{2}}\sin\frac{\theta}{2}|\downarrow\rangle\right)$$

b - global phase; θ, φ - angles on the Bloch sphere.



Functional integral - $\int \mathcal{D}\sigma = \prod_i \int d\theta_i d\varphi_i$

$$Z = \int \mathcal{D}\sigma \, e^{-S}$$

$$S = \int_0^\beta d\tau \, \langle \bar{\sigma} | \partial_\tau + H(\sigma) | \sigma \rangle$$

After taking ∂_τ,

$$\langle \sigma | \partial_\tau | \sigma \rangle = i\left(\dot{b} - \frac{1}{2}\cos\theta\,\dot{\varphi}\right)$$

all $\dot{\theta}$ terms mutually cancel

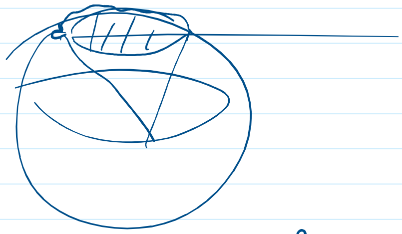
Fix the gauge by choosing $b = \frac{\varphi}{2}$

$$\Rightarrow \langle \sigma | \partial_\tau | \sigma \rangle = \frac{i}{2}(1-\cos\theta)\dot{\varphi}$$

$$S = \underbrace{\int_0^\beta (1-\cos\theta)\,\dot{\varphi}\,d\tau}_{\text{Berry's phase}} + \int_0^\beta H(\sigma)\,d\tau$$

$\int D\theta$ covers all paths on the Bloch sphere.

Berry's phase (geometric phase) is a solid angle subtended by a close path $S(\tau)$, with $S(\beta) = S(0)$.



Whatever is $\hat{H}(\theta)$, we might expect a phase transition when spins interact.

Assuming that there's a phase transition from a "disordered" to an "ordered" phase, where the ordered phase is characterised by some order parameter, Φ.

The effective — G-L — functional is written as

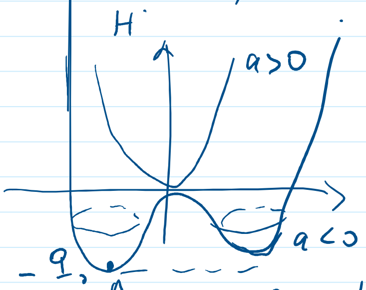
$$\mathcal{H} = \frac{1}{r_0^d}\int d^d x \left[\underbrace{a\Phi^2 + \frac{1}{2}g\Phi^4}_{\text{Landau functional}} + c(\nabla\Phi)^2 \right] \equiv \frac{1}{r_0^d}\int d^d x\, H(\Phi)$$

ultra-violet cut off: e.g. lattice spacing, etc

We've considered this f-l for superconductivity, with a minimal substitution $\nabla \to (\nabla - 2ie)$

Landau: $a \equiv a_0 (T - T_c)$

when $c = 0$, then



$$\frac{\delta \mathcal{H}}{\delta \Phi} = 0 \implies 2a\Phi + 2g\Phi^3 = 0$$

$$\implies \Phi = 0 \quad \text{or} \quad \Phi_0 = -\sqrt{-\frac{a}{g}}$$

for $a < T_c$

a new symmetry breaking minimum (degenerate for $d \geq 2$)

$c \neq 0$ introduces fluctuations around the minimum;

Do the fluctuations destroy PT?

Introduce $\varphi(\underline{r}) = \Phi(\underline{r}) - \Phi_0$

Substituting $\Phi = \Phi_0$ gives $\mathcal{H}_{MF}$

$$\beta\mathcal{H} = \underbrace{\beta\mathcal{H}_0}_{} + \underbrace{\frac{\beta}{r_0^d}\int d^dx\left[\bar{a}\varphi^2 + c(\nabla\varphi)^2 + \frac{1}{2}g\varphi^4\right]}_{\mathcal{K}(\varphi)}$$

Origin of $\bar{a}$: $a\Phi^2 = a(\Phi_0+\varphi)^2 = \underline{a\Phi_0^2} + a\varphi^2$

linear term doesn't contribute by symmetry

$$\frac{1}{2}g\Phi^4 = \frac{g}{2}(\varphi+\Phi_0)^4 = \underline{\frac{g}{2}\Phi_0^4} + 3g\varphi^2\Phi_0^2 - - - -$$

$$\bar{a}\varphi^2 = a\varphi^2 + 3g\varphi^2\Phi_0^2 = \begin{cases} a\varphi^2, & T>T_c \\ -2a\varphi^2, & T<T_c \end{cases}$$

Fluctuational Z:

$$Z = \int \mathcal{D}\varphi\, e^{-\beta\mathcal{K}(\varphi)}$$

If $\mathcal{H} - \mathcal{H}_0 \equiv \mathcal{K}(\varphi)$ is large, the fluctuations are suppressed.

Let's rescale the functional to estimate whether it's large or small;

$$\mathcal{H} = T_c\,\gamma\, F \leftarrow \text{dimensionless}$$

Here F is $\mathcal{H}$ rescaled in such a way that all the terms are of the same order:

rescaling: $x = \lambda\tilde{x}$, $\varphi = \zeta\tilde{\varphi}$

$\tilde{x}$, $\tilde{\varphi}$ ↑ dimensionless all terms of the same order give

$$\lambda^d \bar{a} \xi^2 \sim g \lambda^d \xi^4 \sim c \lambda^{d-2} \xi^2 \sim \gamma T_c r_0^d$$

$$\xi^2 \sim \frac{\bar{a}}{g}, \quad \lambda^2 \sim \frac{c}{\bar{a}}$$

$$\gamma \sim \frac{\lambda^d \bar{a}^2}{g T_c r_0^d} = \left(\frac{c}{\bar{a}}\right)^{\frac{d}{2}} \frac{\bar{a}^2}{g T_c r_0^d}$$

$$= \frac{[a_0 |\tau|]^{2-\frac{d}{2}} c^{\frac{d}{2}}}{g T_c r_0^d}$$

$$= \left(\frac{\lambda}{r_0}\right)^d \frac{a_0^2}{g T_c} |\tau|^{2-\frac{d}{2}}$$

λ plays the role of the "correlation length" at $|\tau| \sim 1$, where $\tau \equiv \frac{T-T_c}{T_c}$

$$\lambda^2 = \frac{c}{a_0}; \quad \xi^2 = \frac{c}{a} \sim \frac{\lambda^2}{|\tau|}$$ – correlation length.

On rescaling,

$$\beta \mathcal{H} \to \gamma F = \gamma \int d^d \tilde{x} \left[\varphi^2 + (\partial_x \varphi)^2 + \frac{1}{2} \varphi^4 \right]$$

For $\gamma \gg 1$, fluctuations are irrelevant.

$$\underline{\gamma \sim \frac{a_0^2}{g T_c} |\tau|^{2-\frac{d}{2}} \gg 1}$$ – Ginzburg-Levanyuk criterion.

For $d \geq 4$, $\gamma \to \infty$ when $\tau \to 0$ – MFT is always applicable.

d is called the upper critical dimension.

For clean superconductivity, $d=3$, $|\tau| \gtrsim 10^{-12}$ for BCS (MF) to work.

What is the low critical dimensionality? For $d < d_{low}$

there's no PT at all.

Let's consider the XY spin model:

$$\mathcal{H}_{XY} = -J \sum_{\substack{\langle ij \rangle \\ \text{next neighbours only}}} \vec{S}_i \cdot \vec{S}_j = -JS^2 \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

All the above estimates are valid.

To see at what low dimensions PTs survive, we consider the correlation function of two spins.

$$K(\underline{R}_i - \underline{R}_j) = \langle \underline{S}_i \cdot \underline{S}_j \rangle \; ; \quad \langle \ldots \rangle = \frac{1}{Z} \int \mathcal{D}\theta \, (\ldots)$$

On a lattice, $\underline{R}_i = a \, \underline{i}$, $\underline{i}$ – an integer value vector on the lattice

Introduce "magnetic" field $\underline{h}$ by adding

$$\mathcal{H}_h = \sum_i \underline{h}_i \cdot \underline{S}_i$$

$$Z_h = \mathrm{Tr}\, e^{-\beta(\mathcal{H} + \mathcal{H}_h)} = \mathrm{Tr}\, e^{-\beta(\mathcal{H}_0 - \sum \underline{h}_i \cdot \underline{S}_i)}$$

Overall magnetisation,

$$\underline{M} \equiv \langle \underline{S} \rangle = \frac{1}{Z} \mathrm{Tr}(\underline{S}_i e^{-\beta \mathcal{H}})$$

$$= -\frac{\delta}{\delta \underline{h}_i} \big(\underbrace{\beta^{-1} \ln Z}_{\mathcal{F}} \big) = -\frac{\partial \mathcal{F}}{\partial h_i}$$

If $\underline{M} \neq 0$ when $\underline{h} = 0$, there's a PT, i.e. spontaneous symmetry breakdown

$S_h = S_{-\underline{h}}$, i.e. $\langle \underline{S}_i \rangle = -\langle \underline{S}_i \rangle = 0$ if the symmetry is unbroken.

Math. meaning is

$$\lim_{N\to\infty} \lim_{h\to 0} \neq \underbrace{\lim_{h\to 0} \lim_{N\to\infty}}_{\text{the limit of interest.}}$$

In terms of h, the correlation f-n

$$K_{\alpha\beta} = -\frac{1}{\beta} \frac{\delta^2 \mathcal{F}}{\delta h_\alpha(\underline{R}_i)\, \delta h_\beta(\underline{R}_j)}$$

$$\underline{h} = \{h_\alpha\} \equiv \{h_x, h_y, h_z\} \text{ etc}$$

In the high-T limit, this is always exp suppressed.

Let's prove this.

$$K_{\alpha\beta}(\underline{R}_1 - \underline{R}_2) = \frac{s^2 \langle \cos(\theta_1 - \theta_2) \rangle}{Z} \quad \Big\| \quad \langle \ldots \rangle = \mathrm{Tr}(\ldots) = \int \mathcal{D}\theta_N = \prod_i \int d\theta_i$$

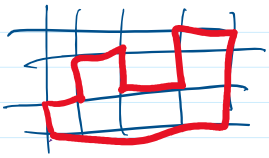
$$= \frac{s^2\, \mathrm{Tr}\left[\cos(\theta_1 - \theta_2)\, e^{\beta J s^2 \sum \cos(\theta_i - \theta_j)}\right]}{\mathrm{Tr}\left[\exp\left(\beta J s^2 \sum \cos(\theta_i - \theta_j)\right)\right]}$$

$$e^{\beta J s^2 \sum_{i\ldots}} = \sum_m \frac{(\beta J s^2)^m}{m!} \left[\sum_{ij} \cos(\theta_i - \theta_j)\right]^m$$

High-T expansion;

$$Z \propto \left\langle \left[\cos(\theta_i - \theta_j)\right]^m \right\rangle = \frac{1}{2^m} \left\langle \left[e^{i(\theta_i - \theta_j)} + e^{-i(\theta_i - \theta_j)}\right]^m \right\rangle$$

It's vanishing due to oscillations, unless all exp are cancelled.



When the path is closed, all exp are cancelled.

In calculating $\dfrac{\langle \cos(\theta_1 - \theta_2) \rangle}{Z}$

All closed path that don't involve θ_1 & θ_2 cancel between the numerator & denominator.

All nonzero contribution are from trajectories that starts on θ_1 and ends on θ_2.

If points 1 & 2 are far apart, we need a rather high degree of expansion of exp to connect θ_1 and θ_2, as only next neighbours contribute in each term.

$$K_{\alpha\beta}(ij) \sim \left(\frac{\beta J S^2}{2}\right)^{\frac{|R_i - R_j|}{a}} \equiv \left(\frac{\beta J S^2}{2}\right)^{m_{min}}$$

$$K(ij) \sim e^{\frac{(R_i - R_j)}{a} \ln \frac{\beta J S^2}{2}} \equiv e^{-\frac{|R_i - R_j|}{\xi}}$$

$$\xi \equiv \frac{a}{\ln \frac{2}{\beta J S^2}} \to \infty \quad \text{when} \quad \beta \to 0.$$

Hence, a PT exists if for $T \to 0$ the dependence of K on $|R_i - R_j|$ is non-exponential.

Let's consider low-T case including quantum corrections:

$$\mathcal{H} = -J S^2 \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) + \sum_j \frac{n_j^2}{2C} + \mathcal{H}_h$$

$$\text{with} \quad n_j = i \frac{\partial}{\partial \theta_j}$$

This model describes, e.g., a set of small superconducting islands of capacity C each connected by tunneling junctions $J_j \sim J$.

We assumed that the order parameter in each island was $\Delta_j = \Delta_0 e^{i\theta_j}$; direction, θ_j, is random.

If all islands are correlated, there's superconductivity. If not — we have the insulator.

$\frac{n_j^2}{2C}$ is the charging energy, with $n_j \propto$ to the # of electrons on the island.

(2e=1 units)

$$\mathcal{E}_{charging} = \frac{q_j^2}{2C} = \underbrace{\frac{C U_j^2}{2} = \frac{C}{2}\dot{\theta}_j^2}_{\text{Josephson relation}}$$

$$S[\{\theta_i\}] = \int_0^\beta d\tau \left\{ \sum_j \frac{C}{2} \dot{\theta}_j^2 - J \sum_{\langle jl \rangle} \cos(\theta_j - \theta_l) \right\}$$

$$Z = \int \mathcal{D}\theta \, e^{-\beta S} \equiv \int \prod d\theta_i \, e^{-\beta S} .$$

Assume that there's MF transition with $\theta_0 \equiv \langle \theta \rangle \neq 0$ is the order parameter.

Introducing $\varphi_j = \theta_j - \theta_0 = \delta\theta_j \rightarrow$ fluctuation,

we have
$$S_{fl} = \int_0^\beta d\tau \left[\frac{C}{2} \sum \dot{\varphi}_j^2 + \frac{J}{2} \sum_{jl} (\varphi_l - \varphi_j)^2 \right]$$

(here cos was expanded to the lowest order in $(\varphi_l - \varphi_j)$).

Let's take Fourier-transform:
$$\varphi_j = \frac{1}{\sqrt{\beta N}} \sum_{\omega_n k} e^{-i\omega_n \tau + i\vec{k}\cdot\vec{r}_j} \varphi_j(i\omega_n, \underline{k})$$

In this terms, using Parseval, we have

$$S_{fl} = \sum_{\omega_n \underline{k}} \left[\frac{C}{2} \omega_n^2 + J\left(1 - \cos(\underline{k}\cdot\underline{m}\, a)\right)\right] \theta(k)\theta(-k$$

$$k \equiv (\omega_n, \mathbf{k})$$

In the long-wave limit, $1-\cos(\mathbf{k}\cdot\mathbf{n}a) \approx \frac{k^2a^2}{2}$

$$\theta_j - \theta_\ell \sim \partial^2\theta$$

Hence, $S_{fl} \sim C \sum_{\omega_n k} \theta(k) \underbrace{\left[\omega_n^2 + v^2k^2\right]}_{G_0^{-1}} \theta(-k)$ collection of SHO

where $\frac{Ja^2k^2}{C} \equiv v^2k^2$

$$G_n \sim \frac{1}{\omega_n^2+v^2k^2}$$

Analytic continuation, $\omega_n \to i\omega$

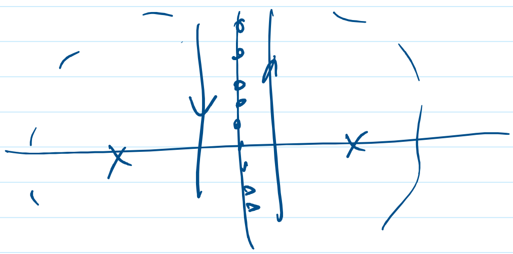
$$G(\omega_n, \mathbf{k}) \to G(\omega, \mathbf{k}) = \frac{1}{\omega^2 - v^2k^2}$$

There are poles at $\omega = \pm vk$ → Goldstone modes

(Always appear as a result of breaking of continuous symmetry)

Now we consider fluctuation contribution by ω_n summation using

$$\beta^{-1}\sum_{\omega_n} f_n = \frac{1}{2\pi i}\oint f(z) \underbrace{\coth\frac{\beta z}{2}}_{\text{counting f-n}}$$



$z = \pm vk = \pm \omega_k$

→ poles contributions.

Hence, after performing the z-integral,

$$\langle \varphi_j^2\rangle_{fl} = \frac{a_0^d}{2C}\int \frac{d^dk}{(2\pi)^d} \frac{\coth\beta\omega_k}{\omega_k}$$

(1) $T \neq 0$; $k \to 0$ and $\beta\omega_k \to 0$,

$\coth \beta\omega_k \to \frac{1}{\beta\omega_k}$

$$\langle \varphi_j^2 \rangle_{fl} \sim \frac{a_0^d T}{2 C v^2} \underbrace{S^{d-1}}_{\text{area of d-dim sphere}} \int \frac{k^{d-1} dk}{k^2}$$

when $d \le 2$, it's divergent at $k \to 0$

$\to$ IR divergence that destroy the MF solution.

Mermin-Wagner theorem – no long-range order for $d \le 2$.

Although LRO is destroyed, PT exists as (we can show) that $K(R_j - R_\ell)$ is non-exponential.

$$K_{\ell j} = \langle \cos(\theta_\ell - \theta_j) \rangle_{fl}$$

$$= \frac{1}{2} \left[\langle e^{i(\varphi_\ell - \varphi_j)} \rangle + c.c. \right]$$

$$= \exp\left[-\frac{1}{2} \langle (\varphi_\ell - \varphi_j)^2 \rangle \right] \quad \text{for}$$

(any Gaussian average).

$$\simeq \frac{a^d}{2C} \int \frac{d^d k}{(2\pi)^d} \frac{1}{\omega_k} \coth \frac{\beta\omega_k}{2} \left[1 - \cos \underline{k} \cdot \underline{R}_{\ell j} \right]$$

It's convergent in the IR limit.

introducing IR cutoff, $|k| > R_0^{-1}$, we have

$$\langle \varphi_j^2 \rangle_{fl} \sim \frac{a_0^d T}{2T} \int_{R_0^{-1}}^{k_c} \frac{k^{d-1}\, dk}{\omega_k^2} \approx \frac{a_0^2 T}{C v^2}$$

$$\xrightarrow[d=2]{} \frac{a_0^d T}{C v^2} \ln k_c R_0$$

$$K_{ej} = e^{-\frac{1}{2}\langle(\varphi_e - \varphi_j)^2\rangle} = \left[|R_e - R_j| \, k_c \right]^{-\frac{T}{2\pi J}}$$

Power-law decay $\equiv$ PT $\equiv$ BKT phase transition.