"Unsharp objectification" revisited - and what does it mean anyway to test quantum uncertainty?

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Quantum measurement: a dialog of big and small

Menu

(1) From Quantum Uncertainty to the Quantum Measurement Problem

(2) From the Insolubility Theorem to Unsharp Objectification

(3) The problem of conceptualising measurement uncertainty

(4) Illustration: Measurement Uncertainty Relations for Qubits

Quantum Uncertainty, Classical Uncertainty, and more

- classical uncertainty $\leftrightarrow$ proper mixture/assemblage of pure states. (Probability distribution on phase space as a mixture of point (Dirac) measures)
- quantum uncertainty $\leftrightarrow$ superposition of pure states.
  - Can be formalised abstractly in so-called operational framework (aka generalised probabilistic models), particularly in quantum logic;
  - not restricted to quantum case
  - Guarantees coincidence of preparation uncertainty and measurement incompatibility.
- more general/alternative forms of uncertainty are possible
  - in the case of a "squit", a state space given as the points of a square, there are pairs of maximally incompatible effects which nevertheless admit common eigenstates (!)

Quantum uncertainty=indeterminacy:
An observable A is **indeterminate** in a pure state (unit Hilbert space vector) φ iff φ is a superposition of at least two eigenstates of A.

This immediately gives rise to the indeterminism, randomness of measurement outcomes: one can only predict the possible outcomes with probabilities less than 1.

Further, this leads to the **quantum measurement problem**: the unitary, hence deterministic evolution axiom for closed systems cannot account for the random occurrence of definite measurement outcomes where the measured observable initially is indeterminate.

Indeterminacy of object observable $\rightarrow$ indeterminacy of pointer observable.

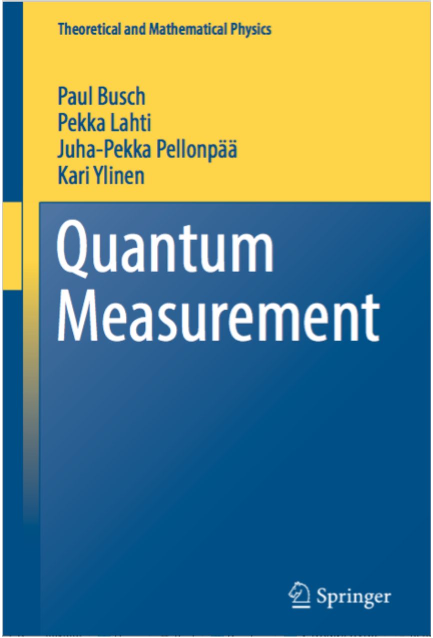
Wigner: can mixed apparatus states help?

Answer: no! $\rightarrow$ *Insolubility Theorem*

Subsequently vastly generalised

– but still contains a possible loophole...

...as made precise in the following book.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The Insolubility theorem

Theorem 22.2 *Let* E *be a (nontrivial) observable of a quantum system* $\mathcal{S}$. *There is no measurement scheme* $(\mathcal{K}, \mathsf{Z}, \sigma, U, g)$ *for* E *that satisfies the pointer value-definiteness condition (22.12) and the pointer mixture condition (22.14) for the whole system* $\mathcal{S}+\mathcal{P}$ *for some (nontrivial) reading scale* $\mathcal{R}$ *and all initial states* ϱ *of* $\mathcal{S}$.

$$Z_i \sigma_i^f = \sigma_i^f \quad \text{for all } i, \tag{22.12}$$

$$U(\varrho \otimes \sigma)U^* = \sum I \otimes Z_i^{1/2}\, U(\varrho \otimes \sigma)U^* I \otimes Z_i^{1/2}. \tag{22.14}$$

Here Z_i are the pointer effects (positive operators with $\sum_i Z_i = I$), $\sigma_i^f = Z_i^{1/2} \sigma^f Z_i^{1/2} / \mathrm{tr}[Z_i \sigma^f]$, where σ^f is the reduced state of the probe (apparatus) in $U(\rho \otimes \sigma)$. ($\mathcal{K}$ is the apparatus/probe Hilbert space, σ the probe's initial state, U a unitary measurement coupling between probe and system, g a suitable scaling function between pointer and observable.)

The RHS of (22.14) represents a state in which the pointer values are definite given (22.12).

(22.14) ensures that the probabilities for the effects E_i of the measured observable E are given by the probabilities for the pointer values (thus ensuring the Born rule).

Dropping (22.12) is the possible loophole: it is not known if the theorem then still holds.
It means that one allows a wider class of pointer observables in the form of POVMs whose effects other than I do not have eigenvalue 1.

If a counterexample could be given in the form of a model, one would have achieved *unsharp objectification*: the desired mixture of states after the measurement with components according to (22.14) in which the pointer is *approximately real/definite*.

This requires us to then show that with such genuinely unsharp pointers one can still *approximately* measure arbitrary sharp observables (since unsharp pointers necessarily only yield measurements of unsharp observables E).

Here we make contact with the theory of approximate measurements of POVMs, that was only recently developed.

The difficulties of conceptualising approximate measurements and appropriately quantifying measurement error in quantum mechanics will be illustrated in the next section, in a review of the history of the uncertainty principle.

Motivation: Heisenberg's Uncertainty Principle (1927)

Heisenberg microscope:

"Let q_1 be the precision with which the value q is known (q_1 is, say, the mean error of q), therefore here the wavelength of the light. Let p_1 be the precision with which the value p is determinable; that is, here, the discontinuous change of p in the Compton effect. Then, according to the elementary laws of the Compton effect p_1 and q_1 stand in the relation

$$p_1\, q_1 \;\sim\; h. \tag{1}$$

- Makes clear reference to error and disturbance
- Sketches proof of *preparation uncertainty relation (PUR)* for the case of a Gaussian (minimum uncertainty) wave function
- Makes informal reference to rms error as standard deviation (of Q-distribution)

Heisenberg 1927: three faces of quantum uncertainty

Preparation Uncertainty (PUR) and Measurement Uncertainty (MUR)

$$(\text{Width of } Q \text{ distribution}) \cdot (\text{Width of } P \text{ distribution}) \sim \hbar$$

$$(\text{Error of } Q \text{ measurement}) \cdot (\text{Error of } P) \sim \hbar$$

$$(\text{Error of } Q \text{ measurement}) \cdot (\text{Disturbance of } P) \sim \hbar$$

PUR – early developments

- Kennard (1927), Weyl (1928) ($p_i = \sqrt{2}\Delta P$, $q_i = \sqrt{2}\Delta Q$)

$$p_i\, q_i \;\geq\; \frac{h}{2\pi}$$

- Robertson (1929)

$$\Delta A\, \Delta B \;\geq\; \frac{1}{2}\left|\langle [A,B]\rangle\right|$$

- Schrödinger (1931)

$$(\Delta A)^2(\Delta B)^2 \;\geq\; \left(\frac{\overline{AB+BA}}{2} - \overline{AB}\right)^2 + \left|\frac{\overline{AB-BA}}{2}\right|^2.$$

Heisenberg 1958 – Physics and Philosophy, Ch. 9

"...in quantum theory the uncertainty relations put a definite limit on the accuracy with which positions and momenta, or time and energy, can be measured simultaneously."

MUR – early denials

- Popper 1934: precursor or EPR (rebuttal by von Weizsäcker)
- Einstein, Podolsky, Rosen (EPR) 1935: use correlations to infer simultaneous sharp values of Q, P
- Park, Margenau 1967: time-of-flight determination of position and momentum
- Aharonov *et al*, since 1990: definite values of incompatible observables *between pre- and post-selection*

MUR – textbook wisdom

- PUR and MUR are conflated; no reflection on how to define measurement error/disturbance, *as conceptually distinct from preparation uncertainty* – hence claim "no joint measurability"
- PUR $\neq$ MUR – hence claim "no limitation on joint measurements"

From sharp incompatibility to approximate joint measurability

… **precise/imprecise joint measurements of noncommuting quantities are impossible/possible.**

Needed:

- notion of imprecise/approximate measurement
- measure of error

MUR – recent challenge

Heisenberg according to Ozawa:

$$\varepsilon(A,\rho)\,\varepsilon(B,\rho) \;\geq\; \tfrac{1}{2}\left|\langle [A,B]\rangle_\rho\right| \quad (???)$$
$$\varepsilon(A,\rho)\,\eta(B,\rho) \;\geq\; \tfrac{1}{2}\left|\langle [A,B]\rangle_\rho\right| \quad (???)$$

- Heisenberg didn't actually state these ... and they is of limited validity
- Ozawa was probably the first to propose explicit formal definitions of measures of error $\varepsilon(A,\rho)$ and disturbance $\eta(B,\rho)$
- Ozawa's correction of the above:

$$\varepsilon(A,\rho)\eta(B,\rho) + \varepsilon(A,\rho)\Delta_\rho B + \Delta_\rho A\eta(B,\rho) \;\geq\; \tfrac{1}{2}\left|\langle [A,B]\rangle_\rho\right|$$

- Quantitatively refined (Branciard), and experimentally confirmed...
- ... yet, conceptually flawed.

MUR – what does the theory (QM) tell us?

Goal: to give suitable measures of approximation error so that the following form of relation can be proven.

$$(\text{combined joint measurement errors for } A, B) \geq (\text{incompatibility of } A, B)$$

Joint Measurability/Compatibility

Definition: joint measurability (compatibility)

Observables $\mathsf{C} = \{C_1, C_2, \ldots, C_m\}, \quad \mathsf{D} = \{D_1, D_2, \ldots, D_n\}$ are *jointly measurable*
if they are margins of an observable $\mathsf{G} = \{G_{k\ell}\}$:

$$C_k = \sum_\ell G_{k\ell}, \quad D_\ell = \sum_k G_{k\ell}$$

Compatibility

Theorem

If one of C, D *is sharp (projection valued), then these observables are jointly measurable iff they commute:*

$$[C_k, D_\ell] = 0$$

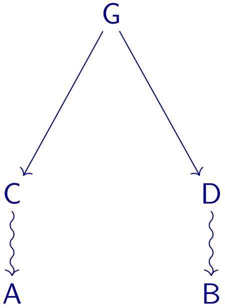
and the joint observable G is uniquely determined: $G_{k\ell} = C_k D_\ell$

compatible & sharp $\Rightarrow$ commuting
compatible & noncommuting $\Rightarrow$ unsharp

Joint measurability in general

Pairs of **unsharp** observables may be jointly measurable
– **even when they do not commute!**

Approximate joint measurement: concept



joint observable

compatible approximating observables

target observables

Task:
(1) formalise compatibility constraint
(2) define suitable measures of *approximation errors*
(3) find (optimal) error bounds

Compatibility

Compatibility of Qubit Effects

Qubit states and observables

$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ (Pauli matrices acting on $\mathbb{C}^2$)

- *States:* $\rho = \frac{1}{2}(I + \boldsymbol{r} \cdot \boldsymbol{\sigma}), \quad |\boldsymbol{r}| \leq 1$
- *Effects:* $A = \frac{1}{2}(a_0 I + \boldsymbol{a} \cdot \boldsymbol{\sigma}) \in [O, I], \quad 0 \leq \frac{1}{2}(a_0 \pm |\boldsymbol{a}|) \leq 1$
- *observables:* $(\Omega = \{+1, -1\})$

$$\mathrm{A}: \ \pm 1 \mapsto A_\pm = \tfrac{1}{2}(I \pm \boldsymbol{a} \cdot \boldsymbol{\sigma}) \quad |\boldsymbol{a}| = 1$$
$$\mathrm{B}: \ \pm 1 \mapsto B_\pm = \tfrac{1}{2}(I \pm \mathbf{b} \cdot \boldsymbol{\sigma}) \quad |\mathbf{b}| = 1$$
$$\mathrm{C}: \ \pm 1 \mapsto C_\pm = \tfrac{1}{2}(1 \pm \gamma)\, I \pm \tfrac{1}{2} \boldsymbol{c} \cdot \boldsymbol{\sigma} \quad |\gamma| + |\boldsymbol{c}| \leq 1$$
$$\mathrm{D}: \ \pm 1 \mapsto D_\pm = \tfrac{1}{2}(1 \pm \delta)\, I \pm \tfrac{1}{2} \boldsymbol{d} \cdot \boldsymbol{\sigma} \quad |\delta| + |\boldsymbol{d}| \leq 1$$

C symmetric (unbiased): $\gamma = 0$

C sharp: $\gamma = 0$, $|\boldsymbol{c}| = 1$; $\quad \rightarrow \quad$ *unsharpness:* $U(\mathrm{C})^2 = 1 - |\boldsymbol{c}|^2$

Compatibility of C, D

Symmetric case (sufficient for optimal compatible approximations):
$C_\pm = \frac{1}{2}(I \pm \boldsymbol{c}\cdot\boldsymbol{\sigma}),\ D_\pm = \frac{1}{2}(I \pm \boldsymbol{d}\cdot\boldsymbol{\sigma})$

Proposition

$\mathsf{C} = \{C_\pm = \frac{1}{2}(I \pm \boldsymbol{c}\cdot\boldsymbol{\sigma})\}$, $\mathsf{D} = \{D_\pm = \frac{1}{2}(I \pm \boldsymbol{d}\cdot\boldsymbol{\sigma})\}$ are compatible if and only if

$$|\boldsymbol{c}+\boldsymbol{d}| + |\boldsymbol{c}-\boldsymbol{d}| \le 2 \qquad (\star)$$

Interpretation: *unsharpness* $U(\mathsf{C})^2 = 1 - |\boldsymbol{c}|^2$; $|\boldsymbol{c}\times\boldsymbol{d}| = 2\|[C_+, D_+]\|$

$$|\boldsymbol{c}+\boldsymbol{d}| + |\boldsymbol{c}-\boldsymbol{d}| \le 2 \ \Leftrightarrow\ \left(1-|\boldsymbol{c}|^2\right)\left(1-|\boldsymbol{d}|^2\right) \ge |\boldsymbol{c}\times\boldsymbol{d}|^2$$

First Main Result

compatible and noncommuting $\Rightarrow$ unsharp

C, D compatible $\ \Leftrightarrow\ U(\mathsf{C})^2 \times U(\mathsf{D})^2 \ge 4\|[C_+, D_+]\|^2$

Qubit compatibility: example

Take $\boldsymbol{c} \perp \boldsymbol{d}$:

$$\mathsf{C}, \mathsf{D} \text{ compatible} \iff |\boldsymbol{c}|^2 + |\boldsymbol{d}|^2 \leq 1 \iff U(\mathsf{C})^2 + U(\mathsf{D})^2 \geq 1$$

$|\boldsymbol{c}| = |\boldsymbol{d}| = \lambda$:

$$C_\pm = \frac{1}{2}(I \pm \boldsymbol{c} \cdot \boldsymbol{\sigma}) = \lambda \frac{1}{2}(I \pm \hat{\boldsymbol{c}} \cdot \boldsymbol{\sigma}) + (1-\lambda)\frac{1}{2} I$$
$$D_\pm = \frac{1}{2}(I \pm \boldsymbol{d} \cdot \boldsymbol{\sigma}) = \lambda \frac{1}{2}(I \pm \hat{\boldsymbol{d}} \cdot \boldsymbol{\sigma}) + (1-\lambda)\frac{1}{2} I$$

C, D compatible iff $\lambda \leq 1/\sqrt{2}$.

Qubit compatibility: example of joint observable

$C_\pm = \frac{1}{2}(I \pm \boldsymbol{c} \cdot \boldsymbol{\sigma})$, $D_\pm = \frac{1}{2}(I \pm \boldsymbol{d} \cdot \boldsymbol{\sigma})$:

$\mathsf{G} = \{G_{k\ell}\}$ is a joint observable, where

$$G_{k\ell} = \frac{1}{4}\left[(1 + k\ell \boldsymbol{c} \cdot \boldsymbol{d})\, I + (k\boldsymbol{c} + \ell \boldsymbol{d}) \cdot \boldsymbol{\sigma}\right], \quad k, \ell \in \{+, -\}.$$

Approximate Joint Measurements for Qubit Observables

Relevant measures of error

Experimental implementability requirement

Any error measure should:

- correctly indicate when a measurement is accurate (error-free)
- ... (insert any other property of error measures you find important)
- ...
- be defined as a quantity that can be estimated directly in terms of an error analysis actually performed in an experiment

Choices of error measures:

distribution comparison error – e.g., distance of observables

calibration error – e.g., error bar width

Value comparison error – only works if target and approximator commute! Hence not suitable for "universal" MURs.

Approximation error for qubits: probabilistic distance

Consider observables A, C on Ω.
Idea: *distribution comparison error*
C is a good approximation to A if $p_\rho^{\mathsf{C}} \approx p_\rho^{\mathsf{A}} \;\; \forall\, \rho$.
Quantify this with some choice of measure of error, e.g., a metric.
Take here: worst-case probability difference

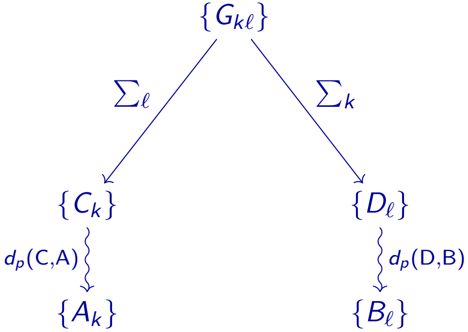
$$d_p(\mathsf{C}, \mathsf{A}) = \sup_\rho \sup_k \big|\mathrm{tr}[\rho C_k] - \mathrm{tr}[\rho A_k]\big| = \sup_k \big\| C_k - A_k \big\|$$

Qubit case: $C_\pm = \frac{1}{2}\big((1 \pm \gamma) I + \boldsymbol{c} \cdot \boldsymbol{\sigma}\big)$, $A_\pm = \frac{1}{2}\big((1 \pm \alpha) I + \boldsymbol{a} \cdot \boldsymbol{\sigma}\big)$

$$d_p(\mathsf{C}, \mathsf{A}) = \big\| C_+ - A_+ \big\| = \tfrac{1}{2}|\gamma - \alpha| + \tfrac{1}{2}|\boldsymbol{c} - \boldsymbol{a}| \;\in\; [0, 1].$$

Measurement Uncertainty Relations for Qubits

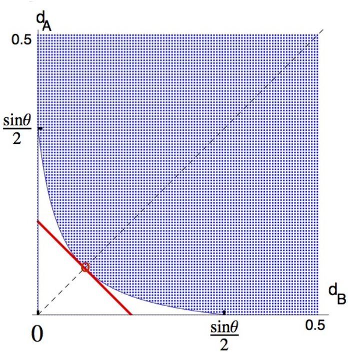
Optimising approximate joint measurements



Goal

To make errors $d_A = d_p(C, A)$, $d_B = d_p(D, B)$ simultaneously as small as possible, *subject to the constraint* that C, D are compatible.

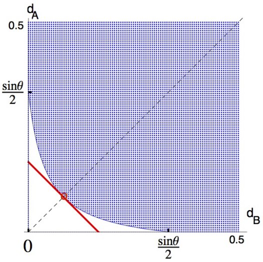
Admissible error region

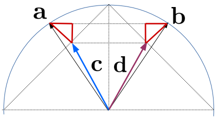


$\sin\theta = |\boldsymbol{a} \times \mathbf{b}|$

$(d_A, d_B) = (d_p(C, A), d_p(D, B)) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}]$ with C, D compatible

trivial approximations: $C_+ = \gamma I$, $D_+ = \delta I$;
then $d_A = \max(\gamma, 1-\gamma) \geq \frac{1}{2}$, $d_B = \max(\delta, 1-\delta) \geq \frac{1}{2}$





$$\sin\theta = |\boldsymbol{a} \times \mathbf{b}|$$

Second Main Result

$$\begin{aligned} |\boldsymbol{c} + \boldsymbol{d}| + |\boldsymbol{c} - \boldsymbol{d}| &\leq 2 \\ U(\mathrm{C})^2 \times U(\mathrm{D})^2 &\geq 4\|[C_+, D_+]\|^2 \\ d_p(\mathrm{C},\mathrm{A}) + d_p(\mathrm{D},\mathrm{B}) &\geq \frac{1}{2\sqrt{2}}\left[\,|\boldsymbol{a} + \mathbf{b}| + |\boldsymbol{a} - \mathbf{b}| - 2\,\right] \end{aligned}$$

$$|\boldsymbol{a} + \mathbf{b}| + |\boldsymbol{a} - \mathbf{b}| = 2\sqrt{1 + |\boldsymbol{a} \times \mathbf{b}|} = 2\sqrt{1 + 2\|[A_+, B_+]\|}$$

Conclusion

Summary

- If *unsharp objectification* works: need to justify the possibility of approximate measurements of sharp observables.
- This requires suitable measures of approximation error between POVMs.
- Can take guidance from recent results on measurement uncertainty relations (MURs)
- MURs can be rigorously formalised and proven (illustrated for qubits).

Outlook

- To explore: measurement models cirumventing the insolubility theorem...
- ...and allowing arbitrarily good approximations of any observable to be measured.

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